# The limit of a function
## section 2.2

## What is the limit of a function?

When we talk about the limit of a function, when $x \to a$ (when $x$ approaches the value of a). It means we want to investigate the behavior of the function $f(x)$ at values very close to a (Remember that Does NOT mean that we are calculating $f(a)$ even though in some cases that you will see soon, $\lim_{x \to a} f(x)$ can become $f(a)$ (Not in general, Just in some cases)

## Mathematical Definition of limit of a function:

Suppose that $f(x)$ is defined when $x$ is near the number a. (this means that f is defined on some open interval that contains a, except possibly at a itself) Then we write

$$\lim_{x \to a} f(x) = L$$

and we say "the limit of $f(x)$, as $x$ approaches a, equals L"

**Example:** Calculate the value of $\lim_{x \to 1} \frac{x-1}{x^2-1}$ ?

**Solution:** We know that function above is not defined at $x=1$, in other words, $x=1$ is not in the domain of the function above (neither $x=-1$)

But when we are talking about limits, we really do Not care about the value of the function at that point, we are only interested in the behaviour of the function at values very close to that specific point.

| $x<1$ | $f(x)$ | $x>1$ | $f(x)$ |
|---|---|---|---|
| 0.5 | 0.666 | 1.5 | 0.4 |
| 0.9 | 0.526 | 1.1 | 0.476 |
| 0.99 | 0.502 | 1.01 | 0.497 |
| 0.999 | 0.5002 | 1.001 | 0.4997 |
| 0.9999 | 0.5000 | 1.0001 | 0.4999 |

we can say

$$\lim_{x \to 1} f(x) = 0.5$$

Example: find the value of $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$ ?

it is clear that the function is NOT defined at $t=0$ but at this point we do NOT care about that. We want to investigate the function at values very close to $t=0$

| $t$ | $f(t)$ |
|---|---|
| ±1.0 | 0.16228 |
| ±0.5 | 0.16553 |
| ±0.1 | 0.16662 |
| ±0.05 | 0.16666 |
| ±0.01 | 0.16667 |
| ±0.0005 | 0.16800 |
| ±0.0001 | 0.2000 |
| ±0.00005 | 0.0000 |
| ±0.00001 | 0.0000 |

if we start from The top and continue our way up to this point We would Think the value of This limit is about 0.16667 which is $\frac{1}{6}$ but if we continue our work to smaller t values, we see the limit goes down to 0 for values close to $t \approx 0$.

This means, Plugging in the # to find the limit of a function is not a good way to find the value of that limit

So we need better ways to calculate the value of limits.

**One sided limit:**

**Definition:** we say the left-hand limit of $f(x)$ as $x$ approaches a (or the limit of $f(x)$ as $x$ approaches a from the left) is equal to L if we can make the value of $f(x)$ arbitrarily close to L by taking $x$ to sufficiently close to a and $x$ less than a, & we write

$$\lim_{x \to a^-} f(x) = L$$

Similarly, if the value of $x$ approaches a from values greater than a, then we get the right-hand limit of $f(x)$ as $x$ approaches a is equal to L and we **Write** $\lim_{x \to a^+} f(x) = L$

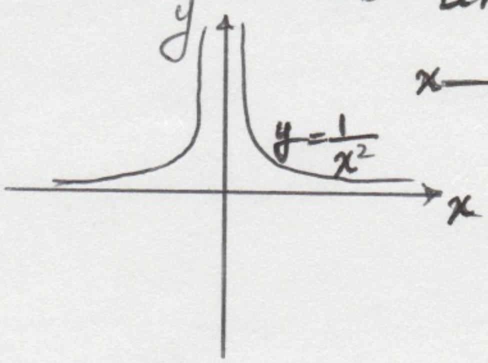
**Very important definition:** $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$ and equal to L

(in other words the limit of a function $f$ exist when $x$ approaches a if and only if the right-hand limit and left-hand limit both is equal to L)

## Infinite Limits:

**Example:** find $\lim_{x \to 0} \frac{1}{x^2}$ if it exist?

**Solution:** as $x$ approaches zero, the function $f(x) = \frac{1}{x^2}$ gets larger and larger. As we see the closer We get to zero, the larger $\frac{1}{x^2}$ would be. This means we never get to a number.




$$\Rightarrow \lim_{x \to 0} \frac{1}{x^2} = \infty$$

(This does NOT mean that we regard $\infty$ as a number, or the function has a limit and it is equal to infinity, it is just saying That the function does not have a limit when $x \to 0$

**Definition:** the line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$\lim_{x \to a} f(x) = \infty$ $\lim_{x \to a^+} f(x) = \infty$ $\lim_{x \to a^-} f(x) = \infty$

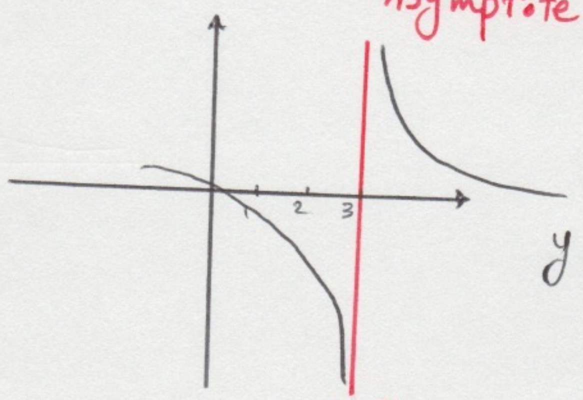
$\lim_{x \to a} f(x) = -\infty$ $\lim_{x \to a^+} f(x) = -\infty$ $\lim_{x \to a^-} f(x) = -\infty$

**Example:** find $\lim\limits_{x \to 3^+} \frac{2x}{x-3}$ and $\lim\limits_{x \to 3^-} \frac{2x}{x-3}$ ?

$\lim\limits_{x \to 3^+} \frac{2x}{x-3} = \infty$ (because when $x$ approaches the value of 3 from values larger than 3, the denominator of the fraction is positive but very small, the numerator is also positive, therefore, the fraction approaches $\infty$)

$\lim\limits_{x \to 3^-} \frac{2x}{x-3} = -\infty$ (why?)

The line of $x=3$ is the vertical Asymptote for the function $\frac{2x}{x-3}$.




$x=3$

$y = \frac{2x}{x-3}$

**Example:** find the vertical asymptote of function $f(x) = \tan(x)$ ?

$\tan(x) = \frac{\sin(x)}{\cos(x)}$ there are potential vertical asymptotes where $\cos(x) \longrightarrow 0$

more accurately $\left\{ \begin{array}{l} \cos(x) \longrightarrow 0^+ \\ \text{when} \\ x \to \frac{\pi}{2}^- \end{array} \right.$ and $\left\{ \begin{array}{l} \cos(x) \longrightarrow 0^- \\ \text{when} \\ x \to \frac{\pi}{2}^+ \end{array} \right.$