Calculating Limits using Limit Laws
Section 2.3

In this section, we use the following properties of limits, called limit laws, to Calculate limits:

Limit Laws: Suppose that c is a constant and $\lim\limits_{x \to a} f(x)$ & $\lim\limits_{x \to a} g(x)$ exist.

① $$\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

② $$\lim_{x \to a} \left[c f(x) \right] = c \lim_{x \to a} f(x)$$

③ $$\lim_{x \to a} \left[f(x) g(x) \right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

④ $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} \quad \left(\text{if } \lim_{x \to a} g(x) \neq 0 \right)$$

⑤ $$\lim_{x \to a} \left[f(x) \right]^n = \left[\lim_{x \to a} f(x) \right]^n \quad (\text{when } n \text{ is a positive integer})$$

⑥ $$\lim_{x \to a} \quad = c$$

⑦ $$\lim_{x \to a} x = a$$

⑧ $$\underset{x \to a}{\text{limit}}\; x^n = a^n \quad (\text{where } n \text{ is a positive integer})$$

(9) $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (where n is a positive integer)

(of coure when n is even, we assume $a > 0$, why?)

(10) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ (where **n** is a positive integer)

(of course when n is even, we assame that $\lim_{x \to a} f(x) > 0$)

Example: Evaluate The following limit? $\lim_{x \to 5} (2x^2 - 3x + 4)$

Solution: $\lim_{x \to 5} (2x^2 - 3x + 4) = \lim_{x \to 5} (2x^2) + \lim_{x \to 5} (-3x) + \lim_{x \to 5} 4$

$= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + 4 = 2(25) - 3(5) + 4 = 39$

Example: Evaluate the following limits? $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} x^3 + 2x^2 - 1}{\lim_{x \to -2} 5 - 3x} = \frac{\lim_{x \to -2} x^3 + \lim_{x \to -2} 2x^2 + \lim_{x \to -2}(-1)}{\lim_{x \to -2} 5 + \lim_{x \to -2} (-3x)}$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{-3 \times (-2)) + 5} = \frac{-8 + 8 - 1}{5 + 6} = -\frac{1}{11}$$

Example: Find $\lim_{x \to 1} \frac{x^2-1}{x-1}$?

Solution: $\frac{x^2-1}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = x+1$

$$\Rightarrow \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} x+1 = 2$$

Example: find $\lim_{x \to 1} g(x)$ where $g(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$?

Pi

Solution: $\lim_{x \to 1} g(x) = \lim_{x \to 1} (x+1) = 2 \rightarrow$ remember that when we are calculating the limit of the above piecewise function, we are trying to investigate function at points very close to 1 <u>Not</u> exactly at $x=1$ so we choose $x+1$ for $g(x)$ rather than π

$$\Rightarrow \lim_{x \to 1} g(x) = \lim_{x \to 1} (x+1) = 2$$

Example: find $\lim_{h \to 0} \frac{(3+h)^2-9}{h}$?

Solution: Suppose $F(h) = \frac{(3+h)^2-9}{h}$

$$\Rightarrow F(h) = \frac{h^2+6h+9-9}{h} = h+6$$

$$\Rightarrow \lim_{h \to 0} F(h) = \lim_{h \to 0} h+6 = 6$$

Example: Find $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$?

$$\frac{\sqrt{t^2+9}-3}{t^2} = \frac{\sqrt{t^2+9}-3}{t^2} \times \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$$

$$= \frac{\cancel{t^2}+\cancel{9}-\cancel{9}}{\cancel{t^2}(\sqrt{t^2+9}+3)} = \frac{1}{\sqrt{t^2+9}+3}$$

$$\Rightarrow \lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{6}$$

Example: show that $\lim_{x \to 0} |x| = 0$?

Solution: recall that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ $\Rightarrow \lim_{x \to 0^+} |x| = \lim x = 0$

$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$

$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 0$

Example: prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist?

Solution: $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$ $\Rightarrow \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

$\lim_{x \to 0^-} \frac{|x|}{x} = \lim \frac{-x}{x} = -1$

Therefore, the limit does Not exist.

Example: $f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8-2x & \text{if } x < 4 \end{cases}$, determine wether $\lim_{x \to 4} f(x)$ exist?

Solution: $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = 0$

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} 8-2x = 0$

Therefore the limit exists and it is equal to $\underline{0}$

Very important

Squeeze Theorem: if $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = \ell$$

Then $\lim_{x \to a} g(x) = \ell$

Example of using Squeeze Theorem (Sandwich Theorem) for calculating limits: Show That $\lim_{x \to 0} (x^2 \sin \frac{1}{x}) = 0$?

solution: First of all we need to realize that we can Not use The following

We can NOT do this: $\lim_{x \to 0} x^2 \sin(\frac{1}{x}) = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$

Because $\lim_{x \to 0} \sin(\frac{1}{x})$ Does Not exist.

But we can say that we know $-1 \leq \sin(\frac{1}{x}) \leq 1$, also the inequality remains true when multiplied by a positive number (in this case x^2)

$$\Rightarrow \quad \underbrace{-x^2}_{f(x)} \leq \underbrace{x^2 \cdot \sin(\tfrac{1}{x})}_{g(x)} \leq \underbrace{x^2}_{h(x)}$$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} -x^2 = 0$

$\lim_{x \to 0} h(x) = \lim_{x \to 0} x^2 = 0$

$\Rightarrow$ According to squeeze theorem,

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 \cdot \sin(\tfrac{1}{x}) = 0$$