We have seen That the limit of function as x approaches a can be often found simply by calculating the value of the function at a ($f(a)$). Functions w/ This property all called continuous at a.

Continuous = without interruption

Formal Definition: A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Three Steps to Test for Continuity:

1) check to see if $f(a)$ is defined (a is in the domain of f)
2) $\lim_{x \to a} f(x)$ exist (left hand limit is equal to the right hand limit)
3) check to see if $\lim_{x \to a} f(x) = f(a)$

IF all the above conditions are satisfied Then the function if is continuous orelse it is discontinuous

Example: where are each of the following functions discontinuous?

a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

$f(2)$ is not defined in the domain of function f, therefore, f is not continuous @ $x=2$

b) $g(t) = \begin{cases} \dfrac{1}{t^2} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$

$f(0)=1$ and defined but $\lim_{t \to 0} \frac{1}{t^2}$ does not exist so $g(t)$ is NOT continuous at $t=0$

This type of discontinuity is called infinite discontinuity

c) $f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Solution: $\lim\limits_{x \to 2} \dfrac{x^2-x-2}{x-2} = \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x+1)}{\cancel{x-2}}$

$= \lim\limits_{x \to 2} x+1 = 3$

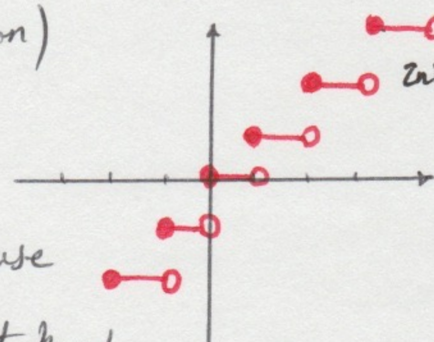
but $f(2) = 1$ (According to the definition)

Therefore $f(x)$ is <u>NOT</u> continuous @ 2

Removable discontinuity because if we change the def. of function at $x=2$ the function becomes continuous

d) $f(x) = [\![x]\!]$ (greatest integer function)

For this function the limit does <u>NOT</u> exist at integerpoints because the left hand limit and the right hand limits are Not equal. (These cases are called Jump discontinuity)



$[x]$ return the integer values that are smaller than or equal to x.

$[5.9] = 5$ $[-3.8] = -4$

$[7] = 7$

$[-4] = -4$

Definition of Continuous from the right and continuous from the left:

* A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

* A function f is continuous from the left at a number a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Example: if you look @ the graph of $f(x) = [\![x]\!]$ is continuous from the right at every integer point but dis continuous from the left because

$$\begin{cases} \lim_{x \to n^-} f(x) = n-1 \neq f(n) & \text{(discontinuous from the left)} \\ \lim_{x \to n^+} f(x) = n = f(n) & \text{(continuous from the right)} \end{cases}$$

Definition: A function f is continuous on an interval if it is continuous at every number in the interval.

Example: Show that function $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval $[-1, 1]$.

Solution: $\lim_{x \to a} f(x) = 1 - \sqrt{1-a^2}$

$$\Rightarrow \lim_{x \to a} f(x) = f(a)$$

Now we have to check to see if the function is continuous from right at -1 and continuous from the left at $+1$

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \quad \& \quad \lim_{x \to +1^-} f(x) = 1 = f(1)$$

Therefore the function is continuous in that interval.

Theorem: if f and g are continuous ~~function~~ functions at $\underline{a}$ and $\underline{c}$ is a constant, then the following functions are also continuous at point a (number a)

$$\underbrace{f+g}, \quad \underbrace{f-g}, \quad \underbrace{cf}, \quad \underbrace{fg}, \quad \underbrace{\frac{f}{g}} \text{ if } g(a) \neq 0$$

These are all Continuous

The Theorem can be proved using limit laws (Refer to page 121 of the book)

Two very important Theorems

a) Any polynomial function is continuous everywhere. That is continuous on $\mathbb{R} = (-\infty, \infty)$

b) Any Rational function is continuous everywhere <u>it is defined</u>, That is, it is continuous on its domain

Example: is function $\dfrac{x^3+2x^2-1}{5-3x}$ continuous?

First we find the domain of this function. Domain $= \{x \mid x \neq \frac{5}{3}\}$

This means, other than $x=\frac{5}{3}$ this function can accept all the values in $\mathbb{R}$. This means except at $x=\frac{5}{3}$ this function is continuous everywhere else.

Theorem: The following types of functions are continuous at every number in their domains:

polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic func.

Example: where is the function $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

This function is composed of three parts $\ln x$, $\tan^{-1} x$ and x^2-1

$\ln x$ is continuous for $x > 0$

$\tan^{-1} x$ is continuous for all x's in $\mathbb{R}$

x^2-1 is a polynomial everywhere in $\mathbb{R}$ but since it is in the denominator of the function f, it can <u>NOT</u> be zero

$$x^2 \neq 1 \Rightarrow x \neq \pm 1$$

Therefor $f(x)$ is continuous where $0 < x < 1$ and $1 < x < \infty$

we show this as $(0, 1)$ and $(1, \infty)$

Example: Evaluate $\displaystyle\lim_{x \to \pi} \frac{\sin(x)}{2 + \cos(x)}$?

this function is continuous except when $2 + \cos(x) = 0 \Rightarrow$ which <u>NEVER</u> happens (why?)

Therefore $\displaystyle\lim_{x \to \pi} \frac{\sin(x)}{2 + \cos(x)} = f(\pi) = \frac{\sin(\pi)}{2 + \cos(\pi)} = \frac{0}{2-1} = 0$

Continuity

Theorem: if f is Continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$.

in other words:

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Example: Evaluate $\lim_{x \to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$?

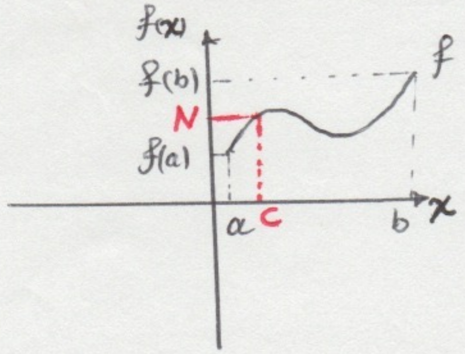
Solution: Since the inverse trigonometric functions are continuous, we can apply the above Theorem.

$$\Rightarrow \lim_{x \to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) = \arcsin\left(\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}\right) = \arcsin\left(\lim_{x \to 1} \frac{\cancel{1-\sqrt{x}}}{(\cancel{1-\sqrt{x}})(1+\sqrt{x})}\right)$$

$$= \arcsin\left(\lim_{x \to 1} \frac{1}{1+\sqrt{x}}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

The Intermediate Value Theorem: (Extremely Important):

if f is a continuous function on the closed interval $[a, b]$ and N is any number between $f(a)$ and $f(b)$ $(f(a) \neq f(b))$ Then there exist a number c in (a, b) interval such That $f(c) = N$.



Example: Does one of the roots of the following equation fall between 1 and 2 ?

$$4x^3 - 6x^2 + 3x - 2 = 0$$

Suppose that $f(x) = 4x^3 - 6x^2 + 3x - 2$ (a polynomial) Therefor $f(x)$ is continnous. Now we calculate $f(1)$ and $f(2)$ $\rightarrow$ $f(1) = -1 < 0$

$f(2) = 12 > 0$

if we pick $N=0$, according to the intermediate value Theorem since $f(1) < N < f(2)$ There is a value between 1 and 2 that $f(N) = 0$ so There is a root between 1 & 2