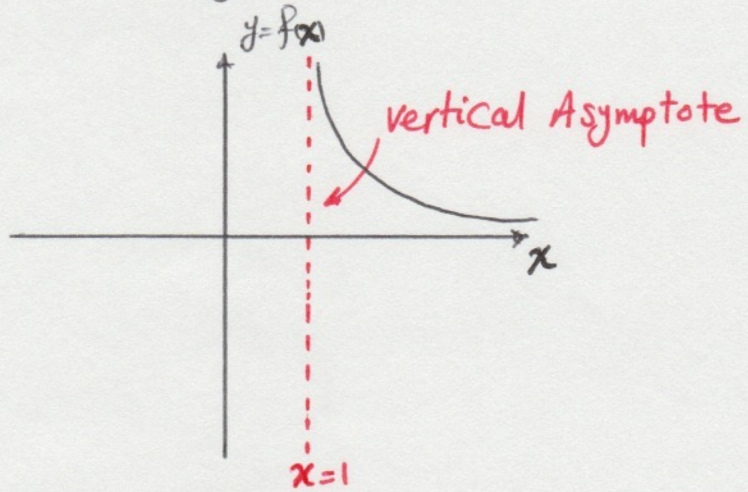
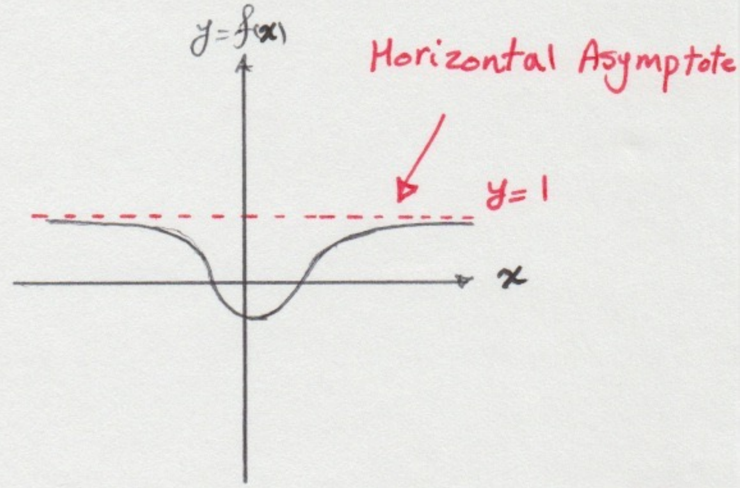
So far we have seen vertical asymptotes. We get them when $f(x)$ approaches infinity (∞) while x approaches a number.

There is a different type of asymptote called <u>horizontal asymptote</u>. We get them when $f(x)$ aproaches a number (L) while x is approaching infinity (∞).





Mathematical Definition of Horizontal Asymptote: The line $y=l$ is called a horizontal asymptote of the curve $y=f(x)$ if either

$$\lim_{x\to\infty} f(x) = l \quad \text{or} \quad \lim_{x\to-\infty} f(x) = l$$

This is the first time that we see the limits such that x approaches infinity instead of a number like <u>a</u>. Therefore, lets define the limits at infinity mathematically, before we go to some examples!

Definition: Let f be a function defined on some interval (a, ∞). Then

$\lim_{x \to \infty} f(x) = l$ means that the value of $f(x)$ can be made arbitrarily close to L by taking $\underline{x}$ sufficiently large.

Definition: Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = l$$

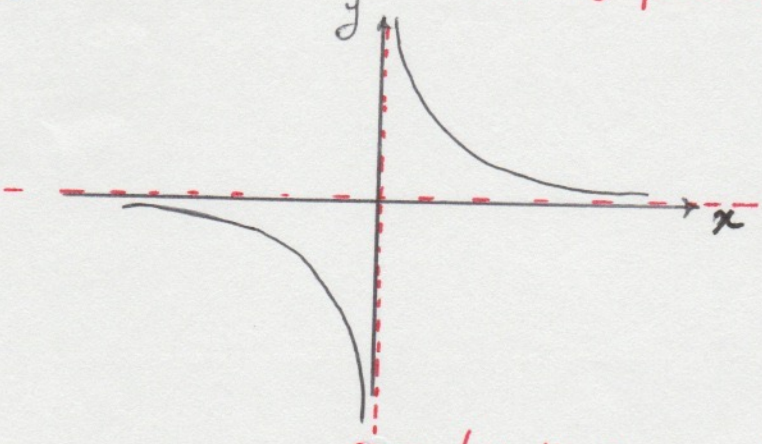
means that the value of $f(x)$ can be made arbitrarily close to l by taking x sufficiently large negative.

Example: find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$?

① $\lim_{x \to \infty} \frac{1}{x} = 0$ because when x is sufficiently large we can make $\frac{1}{x}$ close to $\underline{0}$

② $\lim_{x \to -\infty} \frac{1}{x} = 0$

from ① & ② we can say that $y=0$ (The line) is a horizontal asymptote for $f(x) = \frac{1}{x}$



$y=0$ (Horizontal Asymptote)

$x=0$ (Vertical asymptote because $\lim_{x \to 0^+} \frac{1}{x} = \infty$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

Important Theorem: If $r > 0$ is a rational number then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \quad \& \quad \lim_{x\to -\infty} \frac{1}{x^r} = 0$$

Example: Evaluate $\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2+4x+1}$?

Solution:
$$\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1} = \lim_{x\to\infty} \frac{\frac{3x^2-x-2}{x^2}}{\frac{5x^2+4x+1}{x^2}} = \lim_{x\to\infty} \frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}$$

$$\frac{\lim_{x\to\infty}\left(3-\frac{1}{x}-\frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5+\frac{4}{x}+\frac{1}{x^2}\right)} = \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{1}{x} - 2\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 5 + 4\lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{1}{x^2}} = \frac{3-0-0}{5+0+0} = \frac{3}{5}$$

Example: Compute $\lim_{x\to\infty}\left(\sqrt{x^2+1} - x\right)$?

Both $\sqrt{x^2+1}$ and x are large when x approaches the infinity, therefore it is hard to see the difference b/w $\sqrt{x^2+1}$ and x. in order to solve This problem, we multiply, the numerator and denominator by the <u>Conjugate radical</u>

(Learn This new term)

$$\Rightarrow \sqrt{x^2+1} - x = \left(\sqrt{x^2+1} - x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$= \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} = \frac{1}{\sqrt{x^2+1}+x}$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x^2+1} - x = \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} = 0$$

Example: find the horizontal and vertical asymptotes of the graph of function $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$?

Solution: if we divide both the numerator & the denominator by x we will have:

$$f(x) = \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} \Rightarrow \lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}\sqrt{2+\frac{1}{x^2}}}{\lim_{x\to\infty}\left(3-\frac{5}{x}\right)} = \frac{\sqrt{\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x^2}}}{\lim_{x\to\infty} 3 - 5\lim_{x\to\infty}\frac{1}{x}}$$

$$= \frac{\sqrt{2+0}}{3-5.0} = \frac{\sqrt{2}}{3} \Rightarrow y = \frac{\sqrt{2}}{3} \text{ is the horizontal asymptote}$$

Also $\lim_{x\to-\infty} f(x) = -\dfrac{\sqrt{2}}{3}$ (solve this Part) $\Rightarrow y = -\dfrac{\sqrt{2}}{3}$ is another horizontal asymptote.

This function (its graph also has one vertical asymptote)

$$\lim_{x\to\left(\frac{5}{3}\right)^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty \qquad \lim_{x\to\left(\frac{5}{3}\right)^-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty \Rightarrow x = \frac{5}{3} \text{ is a Vertical Asymptote}$$

Example: Compute $\lim_{x\to\infty}\left(\sqrt{x^2+1} - x\right)$

since both $\sqrt{x^2+1}$ and x become large when x is large, it is hard to see what happens.

$$\lim_{n\to\infty} \sqrt{x^2+1} - x = \lim_{x\to\infty}\left(\sqrt{x^2+1} - x\right)\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x} = \frac{\left(\sqrt{x^2+1}\right)^2 - x^2}{\sqrt{x^2+1} + x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x} = 0$$

Example: evaluate $\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right)$?

Solution: if we assume that $t = \frac{1}{x-2}$ then we know that $t \to \infty$ as x approaches 2^+

$$\lim_{x \to 2^+} \frac{1}{x-2} = \infty \xRightarrow{\text{Therefore}} \lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t \to \infty} \arctan t$$

$$\boxed{= \frac{\pi}{2}}$$

Example: Evaluate $\lim_{x \to 0^-} e^{1/x}$? assume $t = \frac{1}{x} \Rightarrow \lim_{x \to 0^-} t = -\infty$

$$\Rightarrow \lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

Example: Find $\lim_{x \to \infty} \frac{x^2+x}{3-x}$?

Solution: we devide both the numerator and the denominator by x.

$$\Rightarrow \lim_{x \to \infty} \frac{x^2+x}{3-x} = \lim_{x \to \infty} \frac{x+1}{\frac{3}{x}-1} = \frac{\infty}{-1} = -\infty$$

Example: Find $\lim_{x \to \infty} (x^2 - x)$?

it would be wrong to say $\lim_{x \to \infty} (x^2 - x) = \cancel{\lim_{x \to \infty} x^2 - \lim_{x \to \infty} x = \infty - \infty}$

it would be wrong

However we can say $\lim_{x \to \infty} x^2 - x = \lim_{x \to \infty} x(x-1)$ both x and $x-1$ become very large $\Rightarrow \lim_{x \to \infty} x^2 - x = \infty$