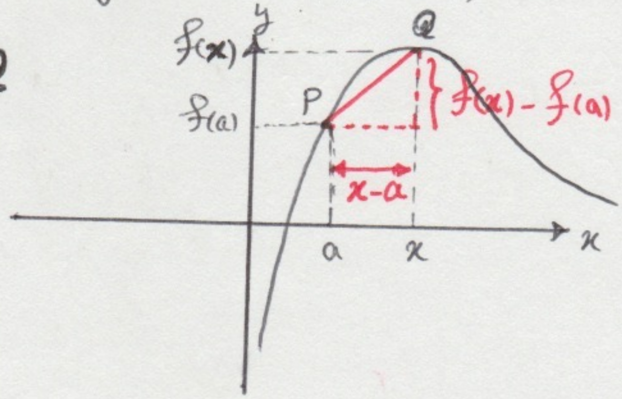
Derivatives and Rates of change   section 2.7

Tangent Line Problem: if a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the line PQ



$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

then we let Q approaches P along the curve c by letting x approach a. if m_{PQ} approaches number m (The limit) exist, then we define tangent line to be the line through P with slope m.

Definition: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope: $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided The the limit exist.

Example: find the equation of the tangent line to the parabola $y = x^2$ at the point $P(1,1)$?

Solution:
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{x^2 - \overset{f(a)}{1}}{x - \underset{a}{1}} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x+1) = 2 \implies \boxed{\underset{\text{slope}}{m} = 2}$$

$$y - y_1 = m(x - x_1) \implies y - 1 = 2(x - \overset{=1}{x_1}) \implies y = 2x - 2 + 1 \implies y = 2x - 1$$

New Definition for tangent line slope:

Some-times, it is easier to calculate the limit when the variable approaches to zero rather than to a non-zero number. With a simple change of variable we can define a new definition for the slope of the tangent line. suppose that we define a new variable h such that

$$\boxed{h = x - a} \quad \text{then} \quad \boxed{x = h + a} \quad \text{therefor} \quad \boxed{f(x) = f(h+a)}$$

if we apply this change of variable to the definition of slope of a tangent line:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{then we get} \quad m = \lim_{h \to 0} \frac{f(h+a) - f(a)}{h}$$

Example: find the equation of the tangent line to the hyperbola $y = \frac{3}{x}$ at the point $(3,1)$?

Solution: $f(x) = \frac{3}{x}$ and we want the slope at $(3,1)$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(h+3) - f(3)}{h} = \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\frac{3-(3+h)}{3+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h} = -\frac{1}{3}$$

therefor the equation for the line is $y - 1 = -\frac{1}{3}(x-3)$

$$3y + x - 6 = 0$$

Velocity Problem: Remember the example of dropping a ball from the balcony of a tower. we have described the average velocity as displacement divided by the time.

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}$$

Now if we are interested in INSTANTANEOUS velocity, we just need to use similar approach that we had for tangent line.

$$\text{instantaneous velocity} = \mathbf{v(a)} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Example: Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground. What is the velocity at $t = 5$ sec, if we know that the distance the ball travels can be calculated using $S(t) = 5t^2$?

$$v(5) = \lim_{h\to 0} \frac{S(h+5)-S(5)}{h} = \lim_{h\to 0} \frac{5(h+5)^2 - 5(5)^2}{h} = \lim_{h\to 0} \frac{5h^2 + 50h + 5\times25 - 5\times25}{h}$$

$$V(5) = \lim_{h\to 0} \frac{5h^2 + 50h}{h} = \lim_{h\to 0} 5h + 50 = 50 \text{ m/s}$$

So far we have talked about $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ but we have never given this a specific name.

Definition of Derivative of a function: the derivative of a function f, at a number a denoted by $f'(a)$, is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \quad \text{if this limit exist.}$$

similarly we can say: $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

Example: find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a?

Solution: $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{(a+h)^2 - 8(a+h) + 9 - (a^2 - 8a + 9)}{h}$

$= \lim_{h \to 0} \frac{\cancel{a^2} + h^2 + 2ah - \cancel{8a} - 8h + \cancel{9} - \cancel{a^2} + \cancel{8a} - \cancel{9}}{h} = \lim_{h \to 0} \frac{h^2 + 2ah - 8h}{h} = \lim_{h \to 0}(h + 2a - 8)$

$= 2a - 8$

Example: find the equation of tangent line to $y = x^2 - 8x + 9$ at the point $(3,-6)$?

from the previous example we know that $f'(a) = 2a - 8$, therefore, the slope of tangent line @ $(3,-6)$ is $f'(3) = 2(3) - 8 = -2 \Rightarrow \boxed{m = -2}$

$y - (-6) = -2(x-3) \Rightarrow \boxed{y = -2x}$

Important Note: the tangent line to $y=f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $\underline{f'(a)}$, or the derivative of $\underline{f}$ at $\underline{a}$.

Derivatives as instantaneous Rate of change:

lets assume That we have a function f such That $y=f(x)$

if we pick a point on this function such as x_1, then the value of the function at That point is $f(x_1)$. Now. if we move from x_1 to a nearby Point x_2 the the value of function changes to $f(x_2)$. in other words a small change from x_1 to x_2 (we call this Δx, read it "delta x") lead to $\Delta y = f(x_2) - f(x_1)$

$$\Delta x = x_2 - x_1$$

$$\Delta y = f(x_2) - f(x_1)$$

Average rate of change of y with respect to change in $x = \dfrac{\Delta y}{\Delta x}$

istantaneous rate of change of y wrt change in x $= \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

wrt: with respect to

Important Note: the derivative $f'(a)$ is the instantaneous rate of change of $y=f(x)$ with respect to (wrt) x when $x=a$.