So far we have considered the derivative of a function f at a fixed number $\underline{a}$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

here we change our point of view and let the number $\underline{a}$ vary. if we replace $\underline{a}$ with variable $\underline{x}$ we get:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if we find this limit, we can consider $f'(x)$ as a new function. f' is called derivative of function f because it has been derived from function f.

Example: if $f(x) = x^3 - x$, find a formula for $f'(x)$?

Solution: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3h^2x + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h} = \lim_{h \to 0} (3x^2 + 3hx + h^2 - 1)$$

$$= 3x^2 - 1 \Rightarrow f'(x) = 3x^2 - 1$$

Example: if $f(x) = \sqrt{x}$, find the derivative of f. Also state the domain of f'?

Solution: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

Domain $= \{x \mid x > 0\}$

Example: find f' if $f(x)=\dfrac{1-x}{2+x}$?

Solution: $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{1-(x+h)}{2+(x+h)}-\dfrac{1-x}{2+x}}{h}$$

$$=\lim_{h\to 0}\frac{(2+x)(x+h-1)-(1-x)(2+x+h)}{h(2+x)(2+h+x)}=\lim_{h\to 0}\frac{2-x-2h-x^2-xh-(2-x+h-x^2-xh)}{h(x+h)(x+h+2)}$$

$$=\lim_{h\to 0}\frac{-3h}{h(2+x+h)(2+x)}=\lim_{h\to 0}\frac{-3}{(2+x)(2+x+h)}=\frac{-3}{(2+x)^2}\Rightarrow f'(x)=\frac{-3}{(2+x)^2}$$

other Notations of $f'(x)$: there are common alternative notations for the derivative of a function as followed:

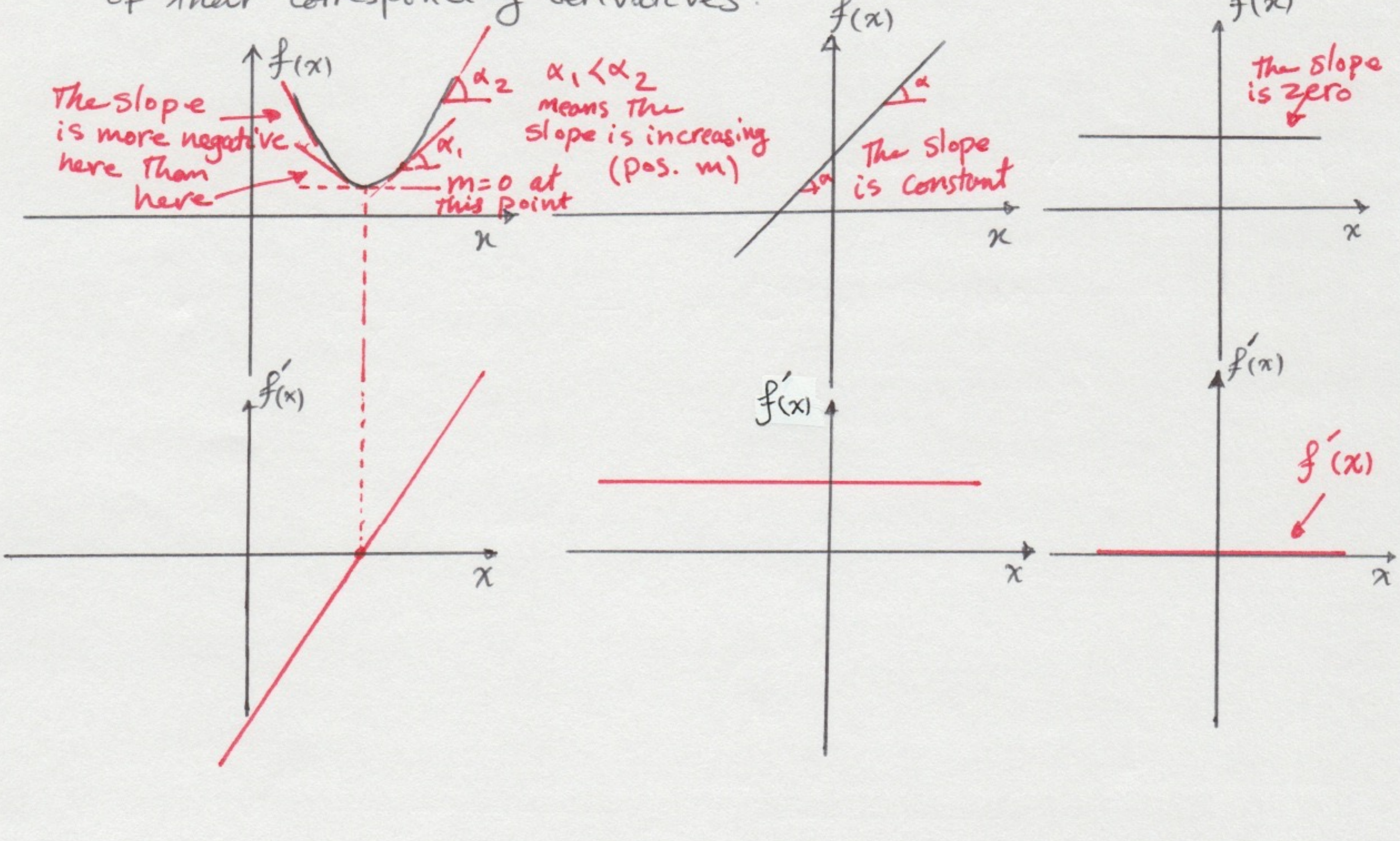
$$f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\frac{d}{dx}f(x)=Df(x)=D_x f(x)$$

Remember That all of the above are the same. D and $\dfrac{d}{dx}$ are called differentiation operation because They indicate the operation of differentiation.

Important Note regarding graph of $f'(x)$:

So far we have seen that if we have a function f, we can obtain f' using the definition of derivatives. On the other hand, we know that the slope of a tangent line on curve $y=f(x)$ at every point of the curve, Represents the derivative of function f at that point. This means, if we have graph of f, we should be able to plot graph of f' from it.

Example: we have the graph of following functions, plot the graph of their corresponding derivatives:



Definition: A function f is differentiable at a if $f'(a)$ exist. It is differentiable on an open interval (a,b), (a,∞), $(-\infty,a)$, $(-\infty,\infty)$ if it is differentiable at every number in the interval.

Example: where is the function $f(x)=|x|$ differentiable?

Solution: we know if $x>0 \Rightarrow |x|=x$, Then if $x+h>0 \Rightarrow |x+h|=x+h$

$$f'(x)=\lim_{h\to 0}\frac{|x+h|-|x|}{h}=\lim_{h\to 0}\frac{(x+h)-x}{h}=1$$

so f is differentiable for all $x>0$

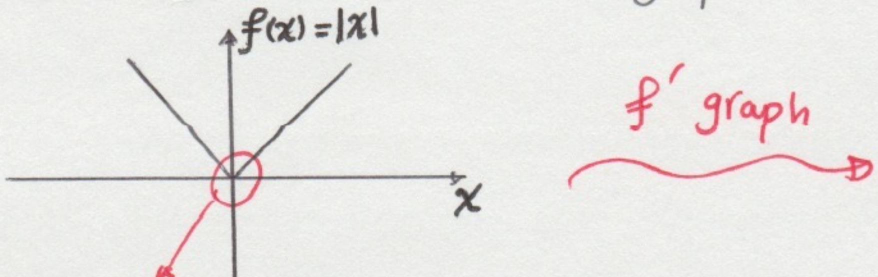
Now, if $x<0 \Rightarrow |x|=-x \Rightarrow |x+h|=-(x+h)$ if $x+h<0$

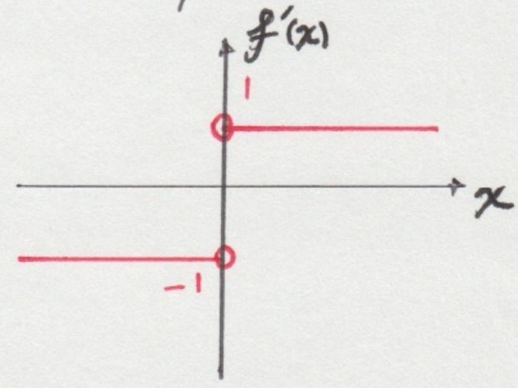
$$f'(x)=\lim_{h\to 0}\frac{|x+h|-|x|}{h}=\lim_{h\to 0}\frac{-(x-h)-(-x)}{h}=\lim_{h\to 0}\frac{-h}{h}=-1$$

Therefore, if $x<0$ Then f is differentiable.

at $x=0$ $\quad \lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} \neq \lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}$

Therefore, f is differentiable everywhere except at $x=0$. we could also see this in the graph.





it is Not Differentiable at $x=0$ (Two different slope at $x=0$)

Theorem: if f is differentiable at $\underline{a}$, Then f is continuous at $\underline{a}$.

(See The proof at page 158 of your book)

Note: The Converse of this theorem is <u>false</u>. That is, there are functions that are continuous but Not differentiable.

e.g. $f(x) = |x|$ is continuous at $x=0$ but it is **Not** differentiable at That point.

How Can a function fail to be differentiable:

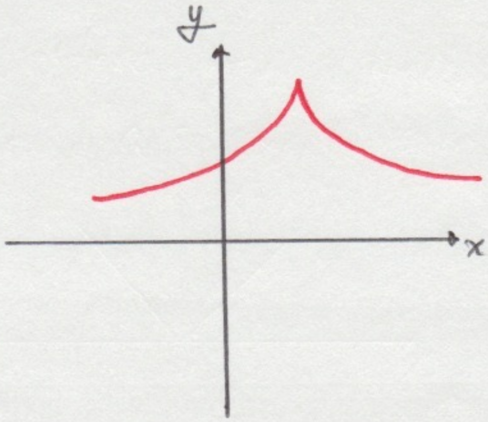
1) if the function has a kink or corner then the graph of function f does Not have tangent at this point

2) the previous Theorem gives another way for a function to fail being differentiable. That is, if the function is Not continuous at that point.

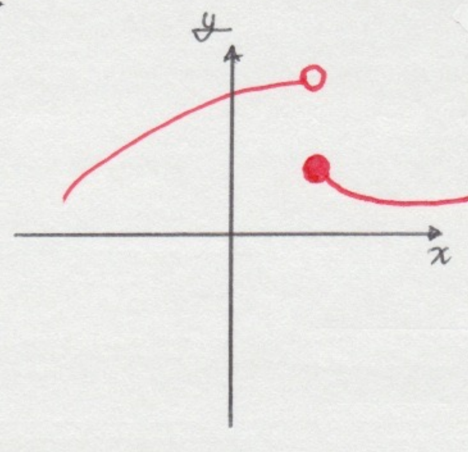
3) The Third posibility is That the curve has a vertical tangent line, so:

$$\lim_{x \to a} |f'(x)| = \infty$$

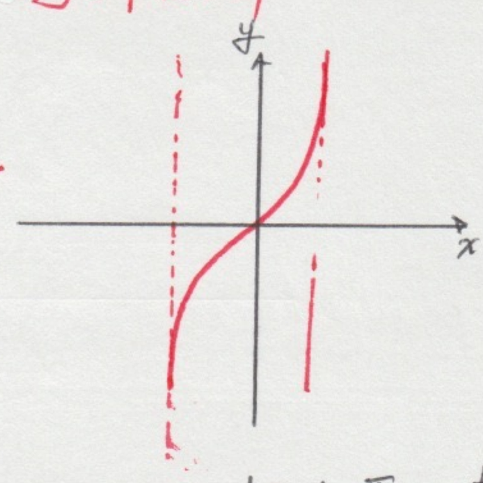
(Remember The concept of Vertical Asymptote)



A kink or a Corner

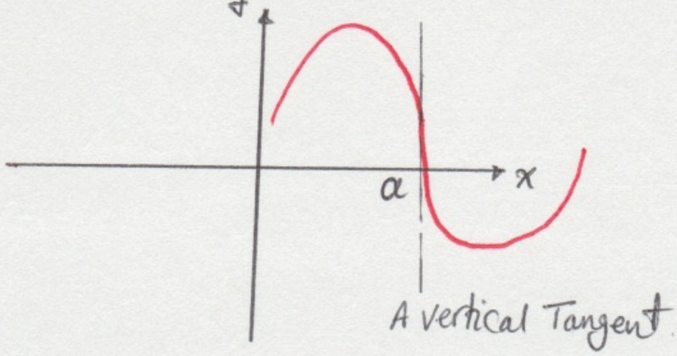


A discontinuity



A vertical Tangent

Another example of A vertical tangent:



Higher Derivatives:

If f is a differentiable function, then its derivative is also a function, so f' may have a derivative of its own denoted by $(f')' = f''$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f.

$$\Rightarrow f''(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2} = \frac{d^2 y}{dx^2}$$

Note where "2," is placed.

Example: If $f(x) = x^3 - x$, find and interpret $f''(x)$?

Solution: from previous examples we know that $f'(x) = 3x^2 - 1$ so the second derivative can be calculated as followed:

$$f''(x) = (f')'(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 1 - [3x^2 - 1]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 3h^2 + 6xh \cancel{-1} \cancel{-3x^2} \cancel{+1}}{h} = \lim_{h \to 0} \frac{3h^2 + 6xh}{h} = 6x$$

second derivative can be considered as the rate of change of a rate of change.

$$f''(x) = 6x$$

higher order derivatives can also be calculated. E.g. the Third derivative can be calculated by taking the derivative of the second derivative and ...

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Example: If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$?

Solution: This question is basically asking for the third & fourth derivative of function f.

we know that $f''(x) = 6x$

$$(f'')'(x) = f'''(x) = \lim_{h \to 0} \frac{6(x+h) - 6x}{h} = \lim_{h \to 0} \frac{6h}{h} = 6 \Rightarrow f'''(x) = 6$$

$$(f''')'(x) = f^{(4)}(x) = \lim_{h \to 0} \frac{6-6}{h} = 0 \Rightarrow f^{(4)}(x) = 0$$

Note: the most familiar example of physical meaning of these derivatives can be given for the motion of an object:

$S(t) \longrightarrow$ equation of movement of an object.

$V(t) = S'(t)$ Velocity

$A(t) = V'(t) = S''(t)$ Acceleration

$J(t) = V''(t) = S'''(t) = A'(t)$ Jerk