Derivatives of Polynomials & Exponential Functions

section 3.1

Introduction: In this section, we learn how to differentiate constant functions, Power functions, Polynomials and exponential functions.

Constant functions: if f is a constant function, such that $f(x) = c$, where c is a constant, it's derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

$\Rightarrow$ **Rule: Derivative of a constant function is always zero**

$$\frac{d}{dx}(c) = 0$$

Power Functions: function f is a power function if $f(x) = x^n$ where n is an integer.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\left[\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)}{2} x^{n-2} h^2 + \cdots + nxh^{n-1} + h^n\right] - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} h + \cdots + nxh^{n-2} + h^{n-1}\right]$$

$$= nx^{n-1}$$

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Rule: If n is any real number ($\mathbb{R}$), then $\frac{d}{dx}(x^n) = nx^{n-1}$

Example: Differentiate (find the derivative) of the following functions:

$f(x) = \frac{1}{x^2}$, $g(x) = \sqrt[3]{x^2}$, $h(t) = t^6$

Solution:

$$f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$$

$$g(x) = \sqrt[3]{x^2} = x^{2/3} \Rightarrow \frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3}\frac{1}{\sqrt[3]{x}}$$

$$h(t) = t^6 \Rightarrow \frac{d}{dt}(t^6) = 6t^{6-1} = 6t^5$$

Example: Find the equation of the tangent line and normal line to the curve $y = x\sqrt{x}$ at point $(1,1)$?

$$y = \sqrt{x^3} = x^{3/2} \Rightarrow \frac{d}{dx}(x^{3/2}) = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x} \Rightarrow f'(x) = \frac{3}{2}\sqrt{x}$$

$$\Rightarrow f'(1) = m = \frac{3}{2}\sqrt{1} = \frac{3}{2} \Rightarrow y - 1 = \frac{3}{2}(x-1) \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$$

→ Tangent Line.

Normal line is perpendicular to the tangent line $\Rightarrow$ $m = \frac{3}{2}$ tangent

Perpendicular: $m' = -\frac{1}{m} = -\frac{2}{3}$

$$\Rightarrow y - 1 = -\frac{2}{3}(x-1) \Rightarrow y = -\frac{2}{3}x + \frac{5}{3}$$

→ Perpendicular line

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Rule: If c is constant and f is a differentiable function, then

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}f(x)$$

proof: let $g(x) = cf(x)$, Then $g'(x) = \lim \dfrac{g(x+h)-g(x)}{h} = \lim \dfrac{cf(x+h)-cf(x)}{h}$

$$= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x)$$

Example: $\dfrac{d}{dx}(3x^4) = 3\dfrac{d}{dx}(x^4) = 3(4x^3) = 12x^3$

$$\frac{d}{dx}(-x) = -\frac{d}{dx}(x) = -1$$

Rule: If f and g are both differentiable, then:

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof: let $8(x) = f(x)+g(x)$

$$8'(x) = \lim\frac{8(x+h)-8(x)}{h} = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)$$

$$\Rightarrow \left[f(x)+g(x)\right]' = f'(x)+g'(x)$$

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Example: find the derivative of function $f(x) = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$

$$\Rightarrow f'(x) = 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

Example: find the point on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal?

$$y' = 4x^3 - 12x \qquad y' = 0$$

Hint: when a line is horizontal, it make $\alpha = 0°$ angle with horizon $\Rightarrow m = \tan\alpha = \tan 0 = 0$
$\Rightarrow m = 0$

$$\Rightarrow 4x^3 - 12x = 0 \Rightarrow x = 0$$

$$\Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

Example: The equation of motion of a particle is $S(t) = 2t^3 - 5t^2 + 3t + 4$, find The equation of acceleration as a function of time?

$$\begin{cases} S'(t) = V(t) = 6t^2 - 10t + 3 \\ S''(t) = V'(t) = A(t) = 12t - 10 \end{cases}$$

Rule: Derivative of the natural exponential function is:

$$\frac{d}{dx}(e^x) = e^x$$

Proof: first lets start with picking $f(x) = a^x$, then $f'(0)$ will be

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \Rightarrow f'(0) = \lim_{h \to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}$$

Now we come back to $f'(x)$ again:

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \Rightarrow f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h} \Rightarrow f'(x) = a^x f'(0)$$

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for $\begin{cases} a=2, \ f'(0) = \lim \dfrac{2^h - 1}{h} \approx 0.69 & \longrightarrow (\text{for } f(x) = 2^x) \\ a=3, \ f'(0) = \lim \dfrac{3^h - 1}{h} \approx 1.10 & \longrightarrow (\text{for } f(x) = 3^x) \end{cases}$

The simplest differentiation formula occurs when $f'(0)=1$ the value of a that lead to $f'(0)=1$ is denoted by $e \approx 2.71$

$$\Rightarrow \begin{cases} e^x = f'(0)e^x \\ f'(0) \text{ when } a=e \text{ is equal to } 1 \end{cases} \Rightarrow \frac{d}{dx}(e^x) = e^x$$

Example: If $f(x) = e^x - x$, find $f'(x)$, $f''(x)$ and $f'''(x)$?

Solution: $f(x) = e^x - x \Rightarrow f'(x) = e^x - 1 \Rightarrow f''(x) = e^x \Rightarrow f'''(x) = e^x$

Example: At what point on curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

$y = e^x \Rightarrow y' = e^x$ (Let's say at point a the tangent line is Parallel to $y=2x$ or $m=2$)

$\Rightarrow e^a = 2 = m \Rightarrow a = \ln 2$