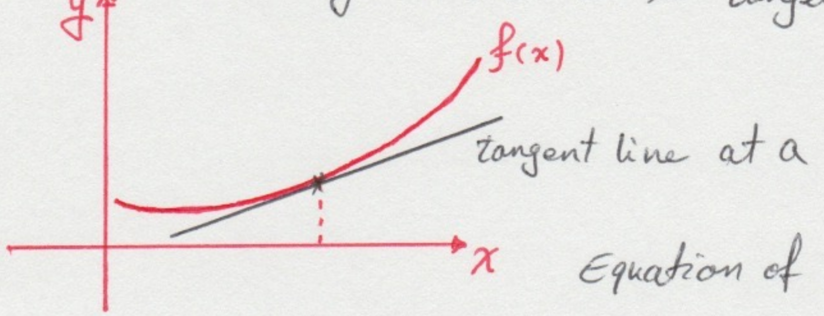
Linear Approximations and Differentials Section 3.10

Suppose That we have graph of a curve like $f(x)$. If we are trying to investigate The behavior of function $f(x)$ at point a, we can do this by appoximating $f(x)$ at a to tangent line at a (and its vicinity).



Equation of tangent line at a : $(a, f(a))$

$$y - f(a) = f'(a)(x-a)$$

At the points close to a, $\underbrace{f(x)}_{\text{Curve}} \approx \underbrace{y}_{\text{Tangent line}}$

$$\Rightarrow \boxed{f(x) \approx f'(a)(x-a) + f(a)}$$

This is called <u>Linear Approximation</u> or <u>Tangent Line Approximation</u>.

it is also called Linearization of f at point a.

Example: find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$?

Solution: $f(x) = \sqrt{x+3} \Rightarrow f(1) = \sqrt{4} = 2$

$$f'(x) = \frac{1}{2\sqrt{x+3}} \Rightarrow f'(1) = \frac{1}{2 \times 2} = \frac{1}{4}$$

$$f(x) = L(x) = f(a) + f'(a)(x-a) \Rightarrow L(x) = 2 + \frac{1}{4}(x-1)$$

$$\Rightarrow \sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4} \quad \text{(when } x \text{ is near 1)}$$

$$\Rightarrow \sqrt{3.98} = \frac{7}{4} + \frac{0.98}{4} = 1.995 \qquad \sqrt{4.05} = \frac{7}{4} + \frac{1.05}{4} = 2.0125$$

Now let's Compare The values from Linearization Theory w/ Actual Values of the function $\sqrt{x+1}$ for different values of x:

	x	$L(x)$	Actual Value
$\sqrt{3.9}$	0.9	1.975	1.9748
$\sqrt{3.98}$	0.98	1.995	1.99499
$\sqrt{4}$	1.0	2	2.0000
$\sqrt{4.05}$	1.05	2.0125	2.01246
$\sqrt{5}$	2	2.25	2.2360
$\sqrt{6}$	3	2.5	2.4494

You see at values Close to 1, $L(x)$ is very close to the Actual value. when we get farther and farther from $x=1$, the approximation becomes Less accurate.

Differentials: The ideas behind linear approximations are sometimes formulated in the terminology of differentials:

suppose: $y = f(x)$ is a differentiable function.

we call dx, differential of the independent variable

dy, differential of the dependent variably

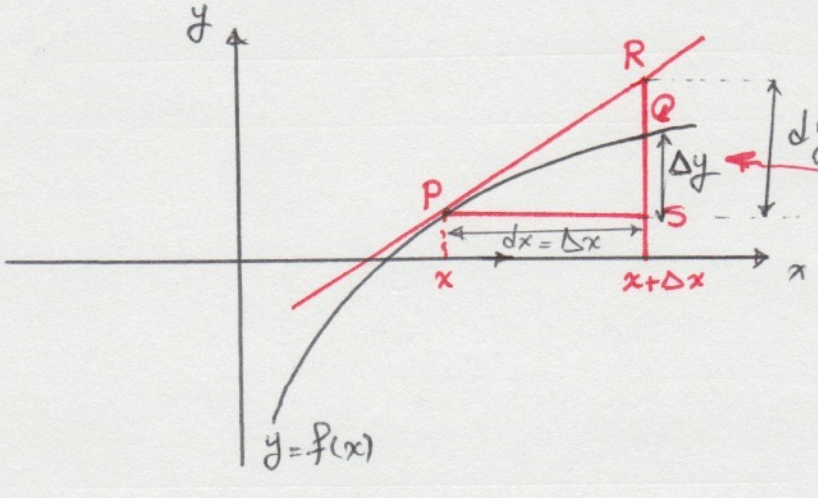
$$dy = f'(x)\,dx$$

Geometric Meaning of Differentials:

if $P(x, f(x))$ and $Q(x+\Delta x, f(x+\Delta x))$ are Two points on the graph of f, let $dx = \Delta x$, Therefore The corresponding change in y is:

$$\Delta y = f(x+\Delta x) - f(x)$$

The slope of The tangent line PR is The derivative of $f'(x)$.
This mean the direct distance between S and R is $f'(x) \cdot dx = dy$
Therefore, dy represents The amount that The tangent line rises or falls (The change in The linearization) where Δy represents the amount That the curve $y = f(x)$ rises or falls when x changes by an amount dx.



Notice The Difference b/w Δy & dy.

Example: Compare the value of Δy and dy if $y=f(x)=x^3+x^2-2x+1$ and
a) x changes from 2 to 2.05 b) x changes from 2 to 2.01 ?

Solution: a) $f(2)=2^3+2^2-2(2)+1=9$

$f(2.05)=(2.05)^3+(2.05)^2-2(2.05)+1=9.717625$

$\Delta y=f(2.05)-f(2)=0.717625$

$dy=f'(x)\,dx=(3x^2+2x-2)\,dx$ we know $x=2$, $x+\Delta x=2.05$ $\Big\}$ $\Delta x=0.05$ $\Rightarrow dx=0.05$

$\Rightarrow dy=\left[3(2)^2+2(2)-2\right]0.05=0.7$

$$\Rightarrow \begin{cases}\Delta y=0.717625\\ dy=0.7\end{cases}$$

b) $f(2.01)=9.140701$

$\Delta y=f(2.01)-f(2)=0.140701$

$\Delta x=dx=0.01 \Rightarrow dy=\left[\left(3(2)^2+2(2)-2\right)\right]0.01=0.14$

As you see when Δx gets smaller, The values of Δy and dy get closer to each other as well.

If we decide to write the linear approximation formula in terms of notation of differentials, we have: $f(x)=f'(a)(x-a)+f(a)$

we replace x w/ $a+dx$ $\Rightarrow f(a+dx)=\underbrace{f'(a)\cdot dx}_{dy}+f(a)$

$\Rightarrow f(a+dx)=f(a)+dy$

going back to the example that we had, $f(x)=\sqrt{x+3}$, we have

$dy=f'(x)\,dx=\dfrac{dx}{2\sqrt{x+3}}$ if $a=1$ and $dx=0.05 \Rightarrow dy=\dfrac{0.05}{2\sqrt{1+3}}=0.0125$

$\Rightarrow \sqrt{4.05}=f(1.05)\approx f(1)+dy=2.0125$

Example: the radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. what is the maximum error in using this value of the radius to compute the volume of the sphere?

Solution: $V = \frac{4}{3}\pi r^3$, lets denote the error in measurement w/ $dr = \Delta r$

Therefore, the corresponding error in volume would be dV

$$dV = 4\pi r^2 dr \quad , \text{w/ } r = 21 \text{ and } dr = 0.05$$

$$\Rightarrow \boxed{\begin{array}{c} dV = 4\pi(21)^2(0.05) \approx 277 \text{ cm}^3 \\ \text{Maximum Error in Calculating Volume} \end{array}}$$

Although this looks like a large error, a better picture of the error is given in terms of <u>relative error</u> which is computed by dividing the error by the total volume.

$$\frac{\Delta V}{V} = \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r}$$

$$\Rightarrow \frac{3dr}{dr} = \text{relative error} = 3 \times \frac{0.05}{21} \approx 0.007 \approx 0.7\%$$

$$\frac{dr}{r} = \text{Percentage error in the radius} = \frac{0.05}{21} \approx 0.0024 = 0.24\%$$