The product and Quotient Rules Section 3.2

In the previous section we saw that $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$ one might get tempted to think that A similar formula might work for multiplication $(f(x) \cdot g(x))$ and division of Two functions $\left(\frac{f(x)}{g(x)}\right)$

But in This chapter we will see that multiplication (product) and division (Quotient) have Two completely different Rules

Rule: If f and g are both differentiable, Then

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

or if u and v are Two functions $(uv)' = u'v + v'u$

proof in page 185 of your book

Example: If $f(x) = xe^{x}$, find $f'(x)$, also find the nth derivative of $f(x)$ or $f^{(n)}x$?

Solution: $u(x) = x$, $v(x) = e^{x}$ $\boxed{(uv)' = u'v + v'u}$

$u(x) = x \Rightarrow u'(x) = 1$

$v(x) = e^{x} \Rightarrow v'(x) = e^{x}$

$$f'(x) = (uv)' \Rightarrow f'(x) = x\,e^{x} + 1 \times e^{x} = (x+1)e^{x}$$

$$f''(x) = (x+1)'e^{x} + (e^{x})'(x+1) = 1 \times e^{x} + e^{x}(x+1) = (x+2)e^{x}$$

$$f'''(x) = (x+2)'e^{x} + (e^{x})'(x+2) = 1 \times e^{x} + e^{x}(x+2) = (x+3)e^{x}$$

$$\vdots$$

$$\Rightarrow f^{(n)}(x) = (x+n)e^{x}$$

Example: Differentiate The function $f(t)=\sqrt{t}\,(a+bt)$?

Solution 1/: $f'(t)= \sqrt{t}\,\frac{d}{dt}(a+bt)+\left(\frac{d}{dt}\sqrt{t}\right)\times(a+bt)$

$$f'(t)= b\sqrt{t}+\left(\frac{d}{dt}(t^{\frac{1}{2}})\right)\left(a+bt\right)= b\sqrt{t}+(a+bt)\left(\tfrac{1}{2}t^{-\frac{1}{2}}\right)$$

$$\Rightarrow f'(t)= b\sqrt{t}+\frac{a+bt}{2\sqrt{t}}=\frac{2bt+a+bt}{2\sqrt{t}}=\frac{a+3bt}{2\sqrt{t}}$$

Solution 2/: Or we can say $f(t)=a\sqrt{t}+bt\sqrt{t}=at^{\frac{1}{2}}+bt^{\frac{3}{2}}$

$$\Rightarrow f'(t)=\frac{a}{2}t^{-\frac{1}{2}}+\frac{3b}{2}t^{\frac{1}{2}}=\frac{a}{2\sqrt{t}}+\frac{3b}{2}\sqrt{t}$$

$$f'(t)=\frac{a+3b\sqrt{t}\cdot\sqrt{t}}{2\sqrt{t}}=\frac{a+3bt}{2\sqrt{t}}$$

Example: if $f(x)=\sqrt{x}\,g(x)$, where $g(4)=2$ and $g'(4)=3$, find $f'(4)$.

$$f'(x)=\frac{d}{dx}(\sqrt{x})\cdot g(x)+\frac{d}{dx}(g(x))\cdot\sqrt{x}$$

$$\Rightarrow f'(x)=\frac{1}{2}x^{-\frac{1}{2}}\cdot g(x)+g'(x)\cdot\sqrt{x}\Rightarrow$$

$$\Rightarrow f'(4)=\frac{1}{2}\frac{1}{\sqrt{4}}\cdot g(4)+g'(4)\sqrt{4}=\frac{1}{2}\times\frac{1}{2}\times 2+3\times 2=6.5$$

The Quotient Rule: If f and g are differentiable, Then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

in other words if $u(x)$ and $v(x)$ are Two functions such That $f(x)=\frac{u(x)}{v(x)}$

$$\Rightarrow f'(x)=\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

Example: let $y=\frac{x^2+x-2}{x^3+6}$, then Calculate y'?

$$y'=\frac{\frac{d}{dx}(x^2+x-2)(x^3+6)-\frac{d}{dx}(x^3+6)\cdot(x^2+x-2)}{(x^3+6)^2}=\frac{(2x+1)(x^3+6)-(3x^2)(x^2+x-2)}{(x^3+6)^2}$$

$$\Rightarrow y'=\frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}$$

Example: Find an equation of the tangent line to the curve $y=\frac{e^x}{1+x^2}$ at the point $M(1,\frac{e}{2})$?

Solution: According to Quotient rule $y'=\frac{(1+x^2)\frac{d}{dx}(e^x)-e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$

$$\Rightarrow y'=\frac{e^x(1+x^2)-e^x(2x)}{(1+x^2)^2}=\frac{e^x(x^2-2x+1)}{(1+x^2)^2}=\frac{e^x(1-x)^2}{(1+x^2)^2}$$

$$\Rightarrow y'(1)=m=\frac{e^1(1-1)^2}{(1+1^2)^2}=0 \Rightarrow \text{the slope is zero}$$

$$\Rightarrow y-\frac{e}{2}=0(x-1)\Rightarrow y=\frac{e}{2}$$ the equation for tangent line

Note: Remember that Not always, all the time that you see a quotient, it is the best idea to use the quotient rule. Sometimes, it is easier to simplify the function that you are taking the derivative and avoid the quotient rule.

Example: if $f(x) = \dfrac{3x^2 + 2\sqrt{x}}{x}$, find $f'(x)$?

$$f(x) = 3x + \frac{2}{\sqrt{x}} = 3x + 2x^{-1/2} \Rightarrow f'(x) = 3 + (-\tfrac{1}{2})(2)\, x^{-\frac{1}{2}-1}$$

$$\Rightarrow f'(x) = 3 - x^{-\frac{3}{2}} = 3 - \frac{1}{\sqrt{x^3}}$$