Important Notes before we go over Derivatives of Trigonometric functions:

1) Review the trigonometric functions (you need to remember formulas such as $\sin(a+b)$, and ...)

2) Remember when we say $f(x)=\sin(x)$, it means sine of the angle whose radian measure is x. (or x is measured in radians)

3) Remember that trigonometric functions are continuous at every # in their domains

Derivative of sine function:

If $f(x)=\sin(x)$, find $f'(x)$?

$\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\left[\frac{\sin x\cos h-\sin x}{h}+\frac{\cos x\sin h}{h}\right]$$

$$f'(x)=\lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right)\right]+\lim_{h\to 0}\frac{\cos x\sin h}{h}$$

$$\lim_{h\to 0}\cos x\left(\frac{\sin h}{h}\right)$$

$$\Rightarrow f'(x)=\sin x\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)+\cos x\lim_{h\to 0}\frac{\sin h}{h}$$

it can be proven that $\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)=0$ & $\lim_{h\to 0}\left(\frac{\sin h}{h}\right)=1$

proof in page 191 & 192 of the book

Therefore, $f'(x) = \sin x \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{0} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{1}$

$f'(x) = \cos x$

Three important rules to remember:

1) $\frac{d}{dx}(\sin x) = \cos x$

2) $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$

*** 3) $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Example: Differentiate $y = x^2 \sin x$?

Solution: $\frac{d}{dx}(y) = \frac{dy}{dx} = \frac{d}{dx}(x^2 \sin x) = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$

$\Rightarrow \underbrace{f'(x) = y' = \frac{dy}{dx}}_{\text{Different Notation of The Same Thing}} = x^2 \cos x + 2x \sin x$

Derivative of Cosine function:

following the similar approach that we had for sine function we can calculate the derivative of cosine function:

$$\boxed{f(x) = \cos x \Rightarrow f'(x) = -\sin x}$$

Now that we know the derivative of sine and cosine functions, we can use Quotient Rule to find the derivative of $\tan x = \dfrac{\sin x}{\cos x}$

$f(x) = \tan x \Rightarrow f'(x) = ?$ $\quad \dfrac{d}{dx}(\tan x) = \dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right)$

$$\Rightarrow f'(x) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{(\cos x)^2} = \frac{\cos x \cdot (\cos x) - \sin x(-\sin x)}{(\cos x)^2}$$

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{(\cos x)^2} = \frac{1}{\cos^2 x} = 1 + \tan^2 x = \sec^2 x$$

Rules for finding derivatives of trigonometric functions:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = 1 + tg^2 x = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example: Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. for what values of x does the graph of f have horizontal tangent?

solution: $$f'(x) = \frac{(1+\tan x)\frac{d}{dx}(\sec x) - \sec x\left(\frac{d}{dx}(1+\tan x)\right)}{(1+\tan x)^2}$$

$$\Rightarrow f'(x) = \frac{(1+\tan x)\sec x \cdot \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2} = \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$\Rightarrow f'(x) = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

Now if $f'(x)=0 \Rightarrow \tan x - 1 = 0 \Rightarrow \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}$

(sec x is never zero) why?

Example: An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at $t=0$. It's position as a function of time is $s(t) = 4\cos t$. find the velocity & acceleration at time t.

$$v(t) = \frac{d}{dt}(s) = \frac{d}{dt}(4\cos t) = 4\frac{d}{dt}(\cos t) = -4\sin t$$

$$A(t) = \frac{dv}{dt} = \frac{d}{dt}(-4\sin t) = -4\frac{d}{dt}(\sin t) = -4\cos t$$

Example: Find 27^{th} derivative of $\cos x$?

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

$f^{(5)}(x) = -\sin x$

$f^{(6)}(x) = -\cos x$

$f^{(7)}(x) = \sin x$

$f^{(27)}(x) = \sin x$

Example: find $\lim\limits_{x\to 0} \frac{\sin 7x}{4x}$? (very Important example)

$$\frac{\sin 7x}{4x} = \frac{\sin 7x}{4x} \times \frac{7}{7} = \frac{7}{4} \cdot \frac{\sin 7x}{7x} \Rightarrow \frac{7}{4}\lim_{x\to 0} \frac{\sin 7x}{7x} = \frac{7}{4}$$

Example: Calculate $\lim\limits_{x\to 0} x\cot x$?

$$\lim_{x\to 0} \frac{x\cos x}{\sin x} = \lim_{x\to 0} \frac{\cos x}{\frac{\sin x}{x}} = \frac{\lim\limits_{x\to 0} \cos x}{\lim\limits_{x\to 0} \frac{\sin x}{x}} = \frac{\cos 0}{1} = 1$$