Suppose that you have $f(x)=\sqrt{x^2+1}$, How do you calculate its derivative?

You can <u>NOT</u> use the formulars that we have learned so far on this problem (if we had $\sqrt{x}$, you could but now a different function is under square root)

One way to solve this problem is to assume $x^2+1=u$ then we have $\left.\begin{array}{l} u(x)=x^2+1 \\ f(x)=\sqrt{u} \end{array}\right\} f(u(x))=\sqrt{x^2+1}$ and use <u>Chain Rule</u> to solve the problem.

The Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F=f\circ g$ define by $F(x)=f(g(x))$ is differentiable at x and F' is given by the product:

$$F'(x)=f'(g(x))\cdot g'(x)$$

In simpler terms (Leibniz notation) $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

$\boxed{\text{Proof in Page 199 \& 204 of your book}}$

Example: Lets go back to the previous example that we started:

Find the derivative of $F(x) = \sqrt{x^2+1}$?

Solution: we can assume $x^2+1 = u$ $\longrightarrow$ means that $f(u) = \sqrt{u}$

According to the chain rule formula $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

$$\frac{du}{dx} = ? \qquad u = x^2+1 \Rightarrow \frac{du}{dx} = 2x$$

$$\frac{df}{du} = ? \quad \frac{d}{du}(\sqrt{u}) = \frac{d}{du}(u^{\frac{1}{2}}) = \frac{1}{2} u^{\frac{1}{2}-1} = \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

$$\Rightarrow \frac{df}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

Example: Differentiate: $y = \sin(x^2)$ and $y = \sin^2 x$?

Solution: (1) we assume that $u = x^2 \Rightarrow y = \sin u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \frac{du}{dx} = \frac{d}{dx}(x^2) = 2x$$

$$\frac{dy}{du} = \frac{d}{du}(\sin u) = \cos u \Rightarrow \frac{dy}{dx} = \cos u \cdot 2x \Rightarrow \frac{dy}{dx} = 2x \cos(x^2)$$

(2) $y = \sin^2 x$, $\sin(x) = u \Rightarrow y = u^2$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du} \Rightarrow \frac{dy}{dx} = \cos x \cdot (2u) \Rightarrow \frac{dy}{dx} = 2 \sin x \cos x$$

The Power Rule Combined with the Chain Rule:

if n is any real number and $u = g(x)$, then $\boxed{\frac{d}{dx}(u^n) = n u^{n-1} \cdot \frac{du}{dx}}$

Lets go back to example 4:

$f(x) = \sqrt{x^2+1} \Rightarrow f'(x) = ?$ $\quad f(x) = (x^2+1)^{\frac{1}{2}}$

$\Rightarrow f'(x) = \frac{1}{2}(x^2+1)^{\frac{1}{2}-1} \times \frac{du}{dx} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$

Example: Differentiate, $y = (x^3-1)^{100}$?

$u = x^3 - 1 \Rightarrow \frac{du}{dx} = 3x^2$ $\quad \frac{dy}{dx} = 100(x^3-1)^{100-1} \cdot \frac{du}{dx}$

$\Rightarrow \frac{dy}{dx} = 100(x^3-1)^{99} \times 3x^2 = 300x^2(x^3-1)^{99}$

Example: find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$

$f(x) = (x^2+x+1)^{-1/3}$ $\quad u = x^2+x+1$ change of variable

$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ $\quad \frac{du}{dx} = 2x+1$

$f(x) = u^{-\frac{1}{3}} \Rightarrow \frac{df}{du} = -\frac{1}{3} u^{-\frac{4}{3}}$

$\Rightarrow \frac{df}{dx} = -\frac{1}{3} u^{-\frac{4}{3}} \cdot (2x+1) = -\frac{1}{3}(2x+1) \frac{1}{\sqrt[3]{(x^2+x+1)^4}}$

Example: find the derivative of function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{d}{dt}\left(\frac{t-2}{2t+1}\right) = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2}$$

$$\Rightarrow g'(t) = \frac{45(t-2)^8}{(2t+1)^{10}}$$

Exercise: Differentiate $y = (2x+1)^5(x^3 - x + 1)^4$?

Answer: $2(2x+1)^4(x^3 - x + 1)^3(17x^3 + 6x^2 - 9x + 3)$

Exercise: Differentiate: $y = e^{\sin x}$

Answer: $y' = e^{\sin x} \cos x$

Another form of Exponential Functions:

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

we know that $a = e^{\ln a}$ (can be written this way)

Why?

$$\Rightarrow f(x) = a^x \Rightarrow f(x) = e^{(\ln a)x}$$

$$\Rightarrow f'(x) = e^{(\ln a)x} \cdot \frac{d}{dx}(\ln a)x = \underbrace{e^{(\ln a)x}}_{a^x} \cdot \ln a$$

$$\Rightarrow f'(x) = a^x \ln a$$

Example: find the derivative of $y = 2^x$?

$$y = 2^x \Rightarrow y' = 2^x \underbrace{\ln 2}_{0.69} \approx 0.69 \cdot 2^x$$

Example: if $f(x) = \sin(\cos(\tan x))$, find $f'(x)$

$$f'(x) = \cos(\cos(\tan x)) \cdot \frac{d}{dx}(\cos(\tan x))$$

$$f'(x) = \cos(\cos(\tan x)) \cdot (-\sin(\tan x)) \frac{d}{dx}(\tan x)$$

$$= -\cos(\cos(\tan x)) \cdot \sin(\tan x)\,(1+\tan^2 x)$$

Example: Differentiate $y = e^{\sec 3\theta}$?

$$\frac{dy}{d\theta} = e^{\sec 3\theta} \cdot \frac{d}{d\theta}(\sec 3\theta) = e^{\sec 3\theta} \cdot \sec 3\theta \ \tan(3\theta) \cdot \frac{d}{d\theta}(3\theta)$$

$$\Rightarrow \frac{dy}{dx} = 3e^{\sec 3\theta} \sec 3\theta \cdot \tan 3\theta$$

Remember we have used the chain Rule 3 times.