so far, all the functions that we have seen, can be described by expressing one variable explicitly in terms of another variable such as $y = \sqrt{x^3+1}$, $y = \sin^2 x$, $y = x \ln x$, ...

But There are some functions that are implicitly defined by a relation between x and y such as

$$x^2 + y^2 = 25 \qquad \text{or} \qquad x^3 + y^3 = 6xy$$

There are numerous cases that we need to differentiate the implicit functions and there are techniques That we can use by applying The chain Rule, to do so.

Example: a) If $x^2+y^2=25$, find $\frac{dy}{dx}$?

b) Find an equation of the tangent line to the circle $x^2+y^2 = 25$ @ Point $M(3,4)$

Solution: a) using chain rule we have $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) \Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$

$$\Rightarrow \frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$\frac{d}{dx}(x^2) = 2x$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = y' = -\frac{x}{y}$$

b) at $M(3,4) \Rightarrow y'(3) = \frac{-3}{4} \Rightarrow m = -\frac{3}{4}$

$$y - 4 = -\frac{3}{4}(x-3) \Rightarrow \boxed{3x + 4y = 25 \quad \text{eq. of tangent line}}$$

Example: find y' if $\sin(x+y) = y^2\cos x$?

Solution: $\frac{d}{dx}\sin(x+y) = \frac{d}{dx}(y^2\cos x) \Rightarrow$ ①

$= \frac{d}{dx}(y^2) \cdot \cos x + \frac{d}{dx}\cos x \cdot y^2 = 2yy'\cos x - y^2\sin x$

$x+y=u$

$\frac{d}{dx}\sin(x+y) = \frac{d}{du}\sin u \cdot \frac{du}{dx}(x+y) = \cos u \cdot (1+y') \Rightarrow \frac{d}{dx}\sin(x+y) = (1+y')\cos(x+y)$ ②

from ① and ② $\Rightarrow 2yy'\cos x - y^2\sin x = (1+y')\cos(x+y)$

$\Rightarrow 2yy'\cos x - y'\cos(x+y) = y^2\sin x + \cos(x+y)$

$\Rightarrow y'(2y\cos x - \cos(x+y)) = y^2\sin x + \cos(x+y)$

$$\Rightarrow y' = \frac{y^2\sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

Example: find y'' if $x^4+y^4=16$?

Solution: $\frac{d}{dx}(x^4+y^4) = \frac{d}{dx}(16) = 0 \Rightarrow 4x^3 + 4y^3y' = 0 \Rightarrow y' = -\frac{x^3}{y^3}$

$(y')' = y'' = -\frac{y^3\frac{d}{dx}(x^3) - x^3\frac{d}{dx}y^3}{(y^3)^2} = -\frac{3x^2y^3 - x^3 \times 3y^2 y'}{y^6}$ → we replace This again w/ $-\frac{x^3}{y^3}$

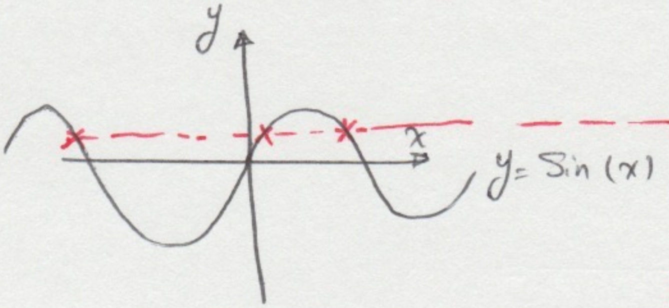
$\Rightarrow y'' = -\frac{3x^2y^3 - 3x^3y^2(-\frac{x^3}{y^3})}{y^6} = -\frac{3x^2y^3 + 3x^6/y}{y^6}$

$\Rightarrow y'' = -\frac{3x^2y^4 + 3x^6}{y^7} = -\frac{3x^2(\overbrace{y^4+x^4}^{16})}{y^7} = -\frac{16 \times 3x^2}{y^7}$

In This part we want to go over derivatives of Inverse of Trigonometric functions. Before, we go over This, we should take a look at Inverse of Trigonometric functions and then Talk about Their derivatives.

Inverse of Trigonometric Functions:

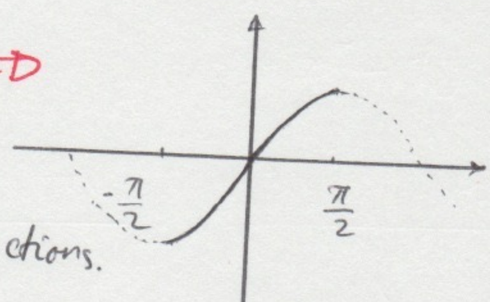
There is one important thing that we need to remember when dealing with inverse of trigonometric functions. Trig. functions are <u>NOT</u> one to one (They do NOT pass the horizontal line test) which means they do Not have inverse functions. We overcome This problem by restricting their domain.



does Not Pass the horizontal line test.

but if we restrict Their domain, say for example for Sin x $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$\Rightarrow$



Now lets go back to The definition of inverse functions.

if f^{-1} is inverse of function f Then $f^{-1}(x) = y \Rightarrow f(f^{-1}(x)) = f(y)$

$$\Rightarrow x = f(y)$$

$$\Rightarrow \boxed{y = \sin^{-1} x \Rightarrow \sin y = x \quad \text{when } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}}$$

Example: Evaluate $\sin^{-1}\left(\frac{1}{2}\right)$ and $\tan\left(\arcsin\frac{1}{3}\right)$?

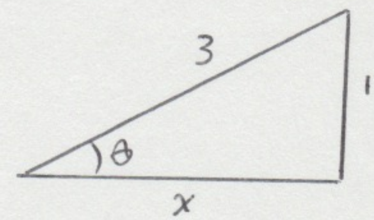
Solution: $\sin^{-1}\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (sine of what value is equal to $\frac{1}{2}$)

$$\sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow \sin^{-1}\left(\sin\frac{\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$\tan\left(\arcsin\frac{1}{3}\right) = ?$

↳ How do we approach This problem? Lets draw a triangle that the sine of one of its angles θ is $\frac{1}{3}$



$$\sin\theta = \frac{1}{3} \Rightarrow \arcsin\frac{1}{3} = \theta$$

$$\Rightarrow \tan\theta = \tan\left(\arcsin\frac{1}{3}\right)$$

According to Pythagorean Theorem $x^2 = 9 - 1 \Rightarrow x = 2\sqrt{2}$

$$\tan\theta = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$\Rightarrow \tan\left(\arcsin\frac{1}{3}\right) = \frac{\sqrt{2}}{4}$$

Cancellation Rules That you need to remember for Sine function:

$\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$\sin\left(\sin^{-1}(x)\right) = x$ for $-1 \leq x \leq +1$

Inverse of Cosine function:

$\cos^{-1} x = y \iff \cos y = x$ and $0 \leqslant y \leqslant \pi$

$\cos^{-1}(\cos x) = x$ for $0 \leqslant x \leqslant \pi$

$\cos(\cos^{-1} x) = x$ for $-1 \leqslant x \leqslant 1$

Inverse of tangent function:

$tg^{-1} x$ or $\tan^{-1} x = y \iff \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Example: simplify the expression $\cos(\tan^{-1} x)$

Solution: suppose $\tan^{-1} x = y \Rightarrow$ we want to find $\cos y$

$$\begin{cases} y = \tan^{-1} x \\ \cos(\tan^{-1}(x)) = \cos y \end{cases}$$

from trigonometric identities we know That $\frac{1}{\cos^2 y} = 1 + \tan^2 y$

$$\Rightarrow \cos^2 y = \frac{1}{1 + \tan^2 y}$$

$$\Rightarrow \cos y = \frac{1}{\sqrt{1 + \tan^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

$y = \tan^{-1} x \Rightarrow \tan y = \tan(\tan^{-1} x) = x$

$\Rightarrow x = \tan y$

Now that we had a discussion on inverse of trigonometric functions, we are read to learn how we can calculate their derivatives:

Let's look at sine function first:

$y = \sin^{-1} x \xrightarrow[\text{means}]{} \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

we take the derivative of $\sin y = x$ implicitly w/ respect to x:

$$x = \sin y \Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}(\sin y) \Rightarrow 1 = \cos y \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$ we know that $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-x^2}$

$$\Rightarrow \boxed{y = \arcsin x \Rightarrow y' = \frac{1}{\sqrt{1-x^2}}}$$

Now suppose $y = \arctan x \Rightarrow \tan y = x$

If we take the derivatives from both sides we have:

$$\frac{d}{dx}(\tan y) = (\tan y)' = (1 + \tan^2 y)\frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\Rightarrow \boxed{\begin{array}{l} y = \arctan x = \tan^{-1} x \\ y' = \dfrac{1}{1+x^2} \end{array}}$$

Example: Differentiate $y = \frac{1}{\sin^{-1} x}$ and $y = x \arctan \sqrt{x}$

$$y = \frac{1}{\sin^{-1} x} \Rightarrow y' = \frac{0 \times \sin^{-1} x - 1 \times \frac{d}{dx}(\sin^{-1} x)}{(\sin^{-1} x)^2} = \frac{-1}{(\sin^{-1} x)^2 \sqrt{1-x^2}}$$

$$y = x \arctan \sqrt{x} \Rightarrow y' = \arctan \sqrt{x} \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\arctan \sqrt{x})$$

$$= \arctan \sqrt{x} + x \frac{1}{1 + (\sqrt{x})^2} \left(\frac{1}{2\sqrt{x}}\right)$$

$$y' = \frac{\sqrt{x}}{2(1+x)} + \arctan \sqrt{x}$$

Remember that you must memorize the following Table:

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$