Derivatives of Logarithmic Functions   Section 3.6

Let $y = \log_a^x$ then $a^y = x$ (According to the definition of Logarithm)

$$\frac{d}{dx}(a^y) = \frac{d}{dx}(x) \Rightarrow \frac{d}{dy}(a^y)\cdot\frac{dy}{dx} = 1 \Rightarrow a^y \ln a \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = y' = \frac{1}{a^y \ln a} = \frac{1}{x \ln a} \Rightarrow \boxed{\begin{array}{l} y = \log_u^x \\ y' = \frac{1}{x \ln a} \end{array}}$$

$$\boxed{\text{or } \frac{d}{dx}\left(\log_a^x = \frac{1}{x \ln a}\right)}$$

if we replace a w/ e, knowing $\ln^e = 1$ Lead to $\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$

Example: Differentiate $y = \ln(x^3 + 1)$?

Solution: let $u = x^3 + 1$, then $y = \ln u$ $\quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \Rightarrow y' = \frac{1}{u}\cdot\frac{du}{dx}$

$$\frac{du}{dx} = 3x^2 \Rightarrow y' = \frac{1}{x^3+1}(3x^2) = \frac{3x^2}{x^3+1}$$

$$\boxed{\text{Remember: } \frac{d}{dx}\ln \underset{u(x)}{u} = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}\left[\ln g(x)\right] = \frac{g'(x)}{g(x)}}$$

Example: find $\frac{d}{dx}\ln(\sin x)$?

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cdot(\cos x) = \cot x$$

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Example: Differentiate $f(x) = \sqrt{\ln x}$? $\quad f(x) = \ln^{\frac{1}{2}} x$

Solution: $f'(x) = \frac{1}{2}(\ln x)^{-\frac{1}{2}} \frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$

Example: Differentiate $f(x) = \log_{10}(2 + \sin x)$?

Solution: $a = 10$, therefore $\longrightarrow$

$$f'(x) = \frac{d}{dx} \log_{10}(2+\sin x)$$

$$f'(x) = \frac{1}{(2+\sin x)\ln 10} \cdot \frac{d}{dx}(2+\sin x)$$

$$\Rightarrow f'(x) = \frac{\cos x}{(2+\sin x)\ln 10}$$

Example: find $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$? $\quad \frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{1}{\frac{x+1}{\sqrt{x-2}}} \cdot \frac{d}{dx}\left(\frac{x+1}{\sqrt{x-2}}\right)$

Solution:

$$\Rightarrow = \frac{\sqrt{x-2}}{x+1} \frac{d}{dx}\left(\frac{x+1}{\sqrt{x-2}}\right) = \frac{\sqrt{x-2}}{x+1} \cdot \frac{1(\sqrt{x-2}) - (x+1)\frac{d}{dx}(\sqrt{x-2})}{(\sqrt{x-2})^2}$$

$$= \frac{\sqrt{x-2}}{x+1} \times \frac{\sqrt{x-2} - (x+1)\frac{1}{2\sqrt{x-2}}}{x-2} = \frac{(x-2) - \frac{1}{2}(x+1)}{(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}$$

Easier Solution: $f(x) = \ln \frac{x+1}{\sqrt{x-2}} = \ln(x+1) - \ln(x-2)^{\frac{1}{2}}$

$$= \ln(x+1) - \frac{1}{2}\ln(x-2)$$

$$\Rightarrow f'(x) = \frac{1}{x+1} - \frac{1}{2} \times \frac{1}{x-2} = \frac{x-5}{2(x+1)(x-2)}$$

Example: find $f'(x)$ if $f(x) = \ln|x|$?

Solution: $f(x) = \begin{cases} \ln x & \text{if } x>0 \\ \ln(-x) & \text{if } x<0 \end{cases} \Rightarrow f'(x) = \begin{cases} \frac{1}{x} & \text{if } x>0 \\ \frac{1}{-x}(-1) & \text{if } x<0 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} \frac{1}{x} & \text{if } x>0 \\ \frac{1}{x} & \text{if } x<0 \end{cases} \Longrightarrow \boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

Logarithmic Differentiation:

To calculate derivatives of complicated functions involving products, quotients, or powers, it is often better that we simplify the function by taking the logarithm first and take the derivative afterward.

Example: Differentiate $y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$?

$$\ln y = \ln\left(\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\right) = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right) = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

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Steps in Logarithmic differentiation:

1- Take the natural log of both sides of an equation $y = f(x)$ and use the laws of logarithms to simplify

2- Differentiate implicitly with respect to x

3- solve the resulting equation for y'

Remember These Two rules:

$$\frac{d}{dx}\left[a^{g(x)}\right] = a^{g(x)} \cdot \ln a \cdot g'(x)$$

$\frac{d}{dx}\left[f(x)^{g(x)}\right]$ (Variable base and Variable exponent): for these cases it is better to use logarithmic differentiation.

Example: Differentiate $y = x^{\sqrt{x}}$?

$$y = x^{\sqrt{x}} \Rightarrow \ln y = \ln\left(x^{\sqrt{x}}\right) = \sqrt{x}\ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sqrt{x}\ln x) \Rightarrow \frac{y'}{y} = \sqrt{x}\frac{d}{dx}(\ln x) + \ln x \cdot \frac{d}{dx}\sqrt{x}$$

$$\frac{y'}{y} = \frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}} = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}$$

$$\Rightarrow \frac{y'}{y} = \frac{2+\ln x}{2\sqrt{x}} \Rightarrow y' = y\left(\frac{2+\ln x}{2\sqrt{x}}\right)$$

$$\Rightarrow y' = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)$$