Exponential Growth and decay   Section 3.8

in many natural phenomena, quantities grow or decay at a rate proportional to their size.

Examples: 1/ Population Growth of bacteria $\rightarrow$ the growth rate (How fast the Population change is proportional to the Population)

2/ Decay of Radioactive materials $\rightarrow$ The rate of mass loss (decay rate of radioactive materials mass is proportional to the mass of the radioactive material)

They both follow $\boxed{\frac{dy}{dt} = ky}$

if $k > 0$ (it is called the Law of natural growth)

if $k < 0$ (it is called the Law of natural decay)

$\rightarrow$ This is Called a <u>Differential Equation</u>.

Because it involves an unknown function y, and its derivative $\frac{dy}{dt}$

What function do we have that its derivative is a constant multiplied by itself? Answer: Exponential Function

> The only solutions of the differential equation $\frac{dy}{dt} = ky$ are the exponential functions.
>
> $$\frac{dy}{dt} = ky \Rightarrow y(t) = y(0)e^{kt}$$

Population Growth:

Suppose that we replace y in the differential equation of previous page by $P(t)$ that represents the population at time t:

$$\Rightarrow \frac{dP(t)}{dt} = k\,P(t) \quad \text{or in short} \quad \frac{1}{P}\cdot\frac{dP}{dt} = k$$

what is the significance of k? $\frac{1}{P}\cdot\frac{dP}{dt}$ is call the **relative growth rate**.

So instead of saying "The growth rate is proportional to population size" we say "The relative growth rate is constant". In other words, constant k represents the relative growth rate.

Example: The world population in 1950 was 2560 millions and 3040 millions in 1960. Model the population in the second half of the 20th century? what is the relative growth rate? use this model to estimate the world's pop. in 1993 & 2020?

Solution: We assume that $t=0$ in 1950. Therefore $P(0) = 2560$ millions & $P(10) = 3040$ millions, since we are assuming $\frac{dP}{dt} = kP$

$$\Rightarrow P(t) = P(0)\,e^{kt} = 2560\,e^{kt}$$

$$P(10) = 3040 \Rightarrow P(10) = 3040 = 2560\,e^{10k} \Rightarrow e^{10k} = \frac{3040}{2560}$$

$$\Rightarrow k \approx 0.017185$$ (The relative growth rate is 1.7% per year)

Therefore, the model is $P(t) = 2560\,e^{0.017185t}$

$$1993 \rightarrow P(43) = 2560 \times e^{43\times 0.017185} \approx 5360 \text{ millions}$$

$$2020 \rightarrow P(70) = 2560 \times e^{70\times 0.017185} \approx 8524 \text{ millions}$$

Radioactive Decay: we know that radioactive substances decay by spontaneously emitting radiation. if $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate $-\frac{1}{m}\cdot\frac{dm}{dt}$ has been found **experimentally** to be constant.

in other words: $\frac{dm(t)}{dt} = -k\,m(t)$

$$\Rightarrow -\frac{1}{m(t)}\cdot\frac{dm(t)}{dt} = k \text{ or in short}$$

$$-\frac{1}{m}\cdot\frac{dm}{dt} = k$$

Therefore, we can show $m(t) = m_0 e^{kt}$ (where k is a negative constant)

Example: half-life of **radium-226** is 1590 years. A sample of 100 mg of this radioactive material exist in the lab.

a) find a formula for the mass of the sample that remains after t years?

b) find the mass after 1000 years?

c) when will the mass be reduced to 30 mg?

Solution: We know that at this moment we have 100 mg of radioactive material.

$\Rightarrow m(0) = 100 \rightarrow m(t) = m(0)\,e^{kt} \Rightarrow m(t) = 100\,e^{kt}$

we know that the half-life of this material is 1590 years. therefore, it takes 1590 years for 100 mg to decay to 50 mg

$$100e^{k \times 1590} = 50 \text{ (become half)} \Rightarrow k = -\frac{\ln 2}{1590}$$

$\Rightarrow m(t) = 100\,e^{-\frac{\ln 2}{1590}\cdot t}$ we could write this as $2^{-t/1590}$

$$\Rightarrow m(t) = 100 \times 2^{\frac{-t}{1590}}$$

b) mass after 1000 years is $m(1000) = 100\, e^{-(\ln 2) \times \frac{1000}{1590}} \approx 65\text{ mg}$

c) we need to find the value of $\underline{\underline{t}}$ when $m(t) = 30$; that is

$$100\, e^{-(\ln 2)t/1590} = 30 \Rightarrow e^{-(\ln 2)t/1590} = 0.3 \Rightarrow -\frac{\ln 2}{1590} \cdot t = \ln 0.3$$

$$\Rightarrow \boxed{t = 2762}$$

Newton's Law of Cooling:

states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings.

$\frac{dT}{dt} = k(T - T_s)$ where k is a constant, T_s is the temperature of surrounding

Example: A bottle of soda at room temperature ($72^\circ F$) is placed in a refrigerator where the temperature is $44^\circ F$. After half an hour the bottle has cooled to $61^\circ F$.

a) what is the temperature of the bottle after another half hour?

b) How long does it take for the soda to cool to $50^\circ F$?

Solution: the temperature of the fridge is $44^\circ F$, therefore $T_S = 44^\circ F$

$\Rightarrow \frac{dT}{dt} = k(T - 44)$ if we let $y = T - 44$ (change of variable)

$$\Rightarrow y(0) = T(0) - 44 = 72 - 44 \Rightarrow y(0) = 28$$

$$y(t) = T(t) - 44 \Rightarrow \frac{dy}{dt} = \frac{dT}{dt} \Rightarrow \frac{dT}{dt} = k(T - 44) \rightsquigarrow \begin{cases} \frac{dy}{dt} = ky \\ y(0) = 28 \end{cases}$$

$y(t) = y(0)\,e^{kt} \Rightarrow y(t) = 28\,e^{kt}$

$T(30) = 61 \Rightarrow y(30) = 61 - 44 = 17 \Rightarrow 28\,e^{30k} = 17 \Rightarrow e^{30k} = \frac{17}{28}$

$k \approx -0.01663$

a) $y(t) = 28\,e^{-0.01663t}$

$\Rightarrow T(t) = 44 + 28\,e^{-0.01663t} \Rightarrow T(60) = 44 + 28\,e^{-0.01663(60)} \approx 54.3^{\circ}F$

b) $T(t) = 50 \Rightarrow 44 + 28\,e^{-0.01663t} = 50 \Rightarrow e^{-0.01663t} = \frac{6}{28}$

$\Rightarrow \boxed{t = 92.6}$

which means the soda cools to 50°F after about one hour and Thirty Three minutes (1:33 hours)

Continuously Compounded Interest:

Example: if a \$1000.00 is invested at 6% interest, compounded annually, then after, 1 year, the investment is worth, $(1000 \times 1.06) = \$1060$, after 2 years its worth $\$[1000 \times (1.06)](1.06) = \$1123.^{60}$ and after t years it worths $\$1000\,(1.06)$

In general, if the interest rate is r, After t years, the intial investment of A_0 is going to become $A_0(1+r)^t$. Usually, interest is compounded more frequently say n times a year. Then, the interest rate in each period is $\frac{r}{n}$ and the value of the investment becomes (**n** times **a year**)

$$A_0\left(1+\frac{r}{n}\right)^{nt}$$

for instance: ($1000 after 3 years become:)

$\$1000(1.06)^3 = \$1191.02$ w/ annual compounding

$\$1000(1.03)^6 = \$1194.05$ w/ Semiannual compounding

$\$1000(1.015)^{12} = \$1195.^{62}$ w/ quarterly compounding

$\$1000\left(1+\frac{0.06}{365}\right)^{365\times 3} = \$1197.^{20}$ w/ daily compounding

As you can see as $n \longrightarrow \infty$ reaches to:

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = \lim_{n\to 0} A_0\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt}$$

$$= A_0\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} = A_0\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$$

$$\Rightarrow A(t) = A_0 e^{rt}$$

if we differentiate: $\frac{dA}{dt} = rA_0e^{rt} = r(A(t))$