Related Rates Section 3.9

In a Related rates problem the idea is to compute the rate of changes of one quantity in terms of the rate of change of another quantity (which can be more easily measured)

The procedure is to find an equation that relates the two quantities and then use the chain Rule to differentiate both sides with respect to (usually time).

The Best way to understand the procedure is by solving some examples:

Example: Air is being pumped into a spherical balloon so its volume is increased at a rate of $100\ cm^3/s$. How fast the radius of the balloon is increasing when the diameter is 50 cm?

Solution: The rate of increase of volume is $100\ cm^3/s$ $\longrightarrow \frac{dV}{dt} = 100\ cm^3/s$

we are interested in the rate of increase of diameter (or radius) or $\frac{dr}{dt}$

we know that Volume of a sphere $= \frac{4}{3}\pi r^3$

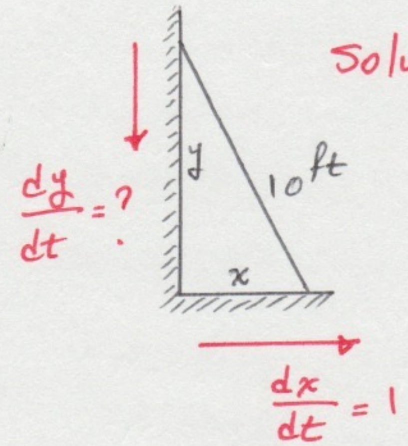
$\Rightarrow V = \frac{4}{3}\pi r^3$ from chain rule: $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$$\Rightarrow \frac{dV}{dr} = 4\pi r^2 \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi (25)^2} \cdot 100 = \frac{1}{25\pi} \approx 0.0127\ cm/s$$

$$\Rightarrow \boxed{\frac{dr}{dt} = 0.0127\ cm/s}$$

Example: A ladder <u>10</u> ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall, at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



Solution: we know that $\frac{dx}{dt} = 1$ ft/s

we know the ladder is 10 ft long and we are interested in $\frac{dy}{dt}$ when $x = 6$ ft

Pythagorean Theorem gives $x^2 + y^2 = 10^2$

$$6^2 + y^2 = 10^2 \Rightarrow \boxed{y = 8 \text{ ft}}$$

which means that when $x = 6$ ft, the height of the top of the ladder is 8 ft.

we also know $x^2 + y^2 = 100$

$$\Rightarrow \frac{d}{d}(x^2) + \frac{d}{dt}(y^2) = 0 \Rightarrow \text{Chain Rule: } 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

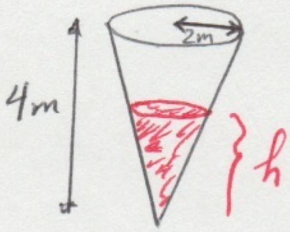
when $x = 6$, $\frac{dx}{dt} = 1$, & $\frac{dy}{dt} = ?$
$y = 8$

$$\frac{dy}{dt} = -\frac{6}{8} \times 1 = -\frac{3}{4} \text{ ft/s}$$

$$\Rightarrow \boxed{\frac{dy}{dt} = -\frac{3}{4} \text{ ft/s}}$$

The negative sign means that the distance between the top of the ladder and ground is <u>Decreasing</u>

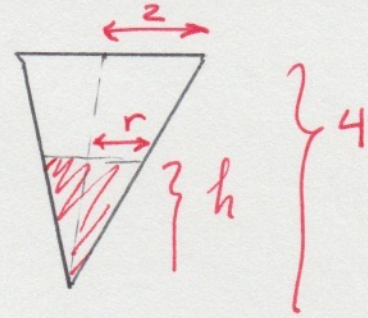
Example: A water tank has the shape of an inverted circular cone with the base radius of 2m and height of 4m. If water is being pumped into the tank at the rate of $2\ m^3/min$, find the rate at which the water level is rising when water is <u>3m</u> deep?



Volume of the Cone $= \frac{1}{3}\pi r^2 h$

Right now function V has been expressed in terms of Two Variables h, and r. it is useful to eliminate one.

$\Rightarrow$ We use similar triangles in this figure:



$$\frac{r}{h} = \frac{2}{4} \Rightarrow r = \frac{h}{2}$$

$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi}{12}h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{4}\pi h^2 \cdot \frac{dh}{dt}$$

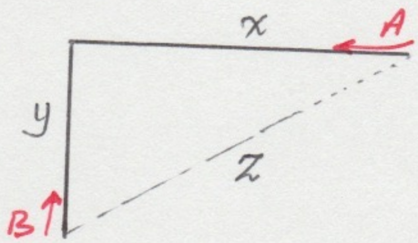
$$\Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt} \Rightarrow h = 3\ m \quad \text{and} \quad \frac{dV}{dt} = 2m^3$$

$$\Rightarrow \frac{dh}{dt} = \frac{4}{\pi(3)^2} \cdot 2 = \frac{8}{9\pi}$$

$$\frac{dh}{dt} \approx 0.28\ m/min$$

Example: Car A is traveling west at 50 mi/h and Car B is traveling north at 60 mi/h. Both cars headed for the intersection of the two roads. At what rate are the cars approaching each other when A is 0.3 miles and car B is 0.4 miles from the intersection?

Solution:



$$z^2 = x^2 + y^2$$

we have $\frac{dx}{dt} = -50$ mi/h $\qquad \frac{dy}{dt} = -60$ mi/h

$\frac{dz}{dt}$ when $x = 0.3$ & $y = 0.4$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

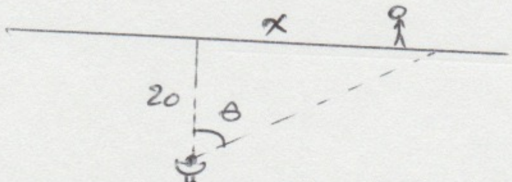
when $x = 0.3$, $y = 0.4 \Rightarrow z = 0.5 \qquad (x^2+y^2=z^2)$

$$\Rightarrow 2\times 0.5 . \frac{dz}{dt} = 2\times 0.3\times(-50) + 2\times 0.4(-60)$$

$$\frac{dz}{dt} = -0.6\times 50 - 0.8\times 60 = -30 - 48 = -78 \text{ mi/h}$$

$$\Rightarrow \frac{dz}{dt} = -78 \text{ mi/h}$$

Example: A Man walks along a straight path at a speed of 4 ft/s. A search light is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the search light rotating when the man is 15 ft from the point on the path closest to the search light?



$$\frac{dx}{dt} = 4 \text{ ft/s}$$

the distance from the linght to the person = 25 ft

$\tan\theta = \frac{x}{20} \Rightarrow x = 20\tan\theta$

why? $\rightarrow 15^2 + 20^2 = (\text{distance})^2 = 625 \Rightarrow \frac{dx}{dt} = 20 \times (1+\tan^2\theta) \cdot \frac{d\theta}{dt}$

$\Rightarrow$ distance = 25

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{20} \cdot \frac{dx}{dt} \times \frac{1}{1+\tan^2\theta} = \frac{1}{20} \times 4 \times \cos^2\theta$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{5}\cos^2\theta$$

when $x = 15 \longrightarrow$ Distance = 25

$$\Rightarrow \cos\theta = \frac{20}{25} = \frac{4}{5}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{5} \times \left(\frac{4}{5}\right)^2 = \frac{16}{125}$$

$$\Rightarrow \frac{d\theta}{dt} = 0.128 \text{ rad/s}$$