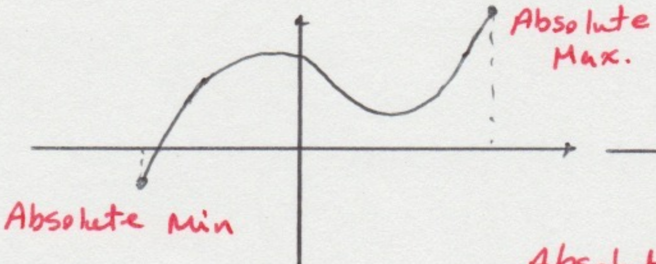
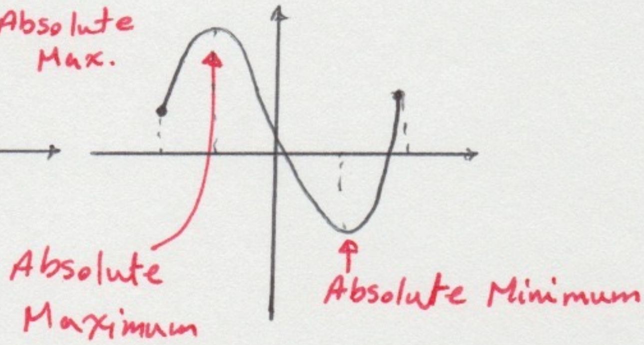
Definition: Let c be a number in the Domain D of a function f. Then

$f(c)$ is : **Absolute Maximum** value of f on D if $f(c) \geq f(x)$ for all x in D

Absolute Minimum Value of f on D if $f(c) \leq f(x)$ for all x in D





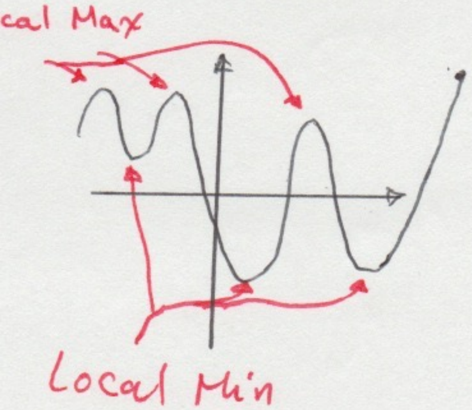
Note: sometimes absolute Max or Min are called <u>global</u> **Max** or <u>global</u> **Min**

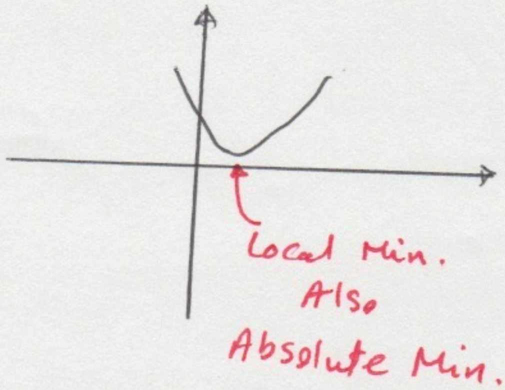
> **These points are Called Extreme Values of f**

Definition: The number $f(c)$ is a:

Local Maximum value of f if $f(c) \geq f(x)$ when x is near c

Local Minimum value of f if $f(c) \leq f(x)$ when x is near c



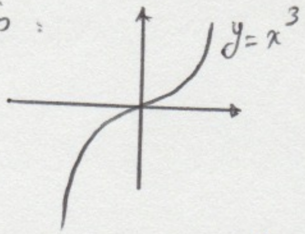


Example: $f(x) = \cos x \longrightarrow$ Local & absolute Maximum Values $= 1$

Local & absolute Minimum Vales $= -1$

Example. $f(x) = x^2 \longrightarrow x^2 \geqslant 0$ therefore $f(0) = 0$ is the local & absolute Minimum. There is no highest point for $f(x) = x^2$ (goes to infinity) so this function Does Not have any Maximum **Value** (neither local, **Nor** global)

Example: $f(x) = x^3$ if we plot this:

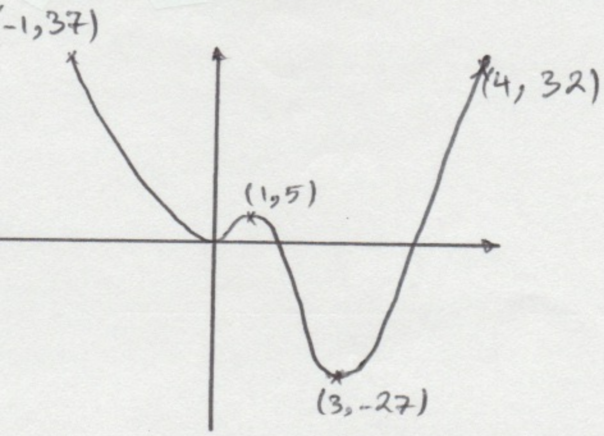


There is no absolute maximum or minimum. **there is No local Max or Min either**

Fermat's Theorem: If f has a Local Max or Local Min at c, and if $f'(c)$ exists, then $f'(c) = 0$

The fact that $f'(x) = 0$ at $(0,0)$, only means that we have a tangent line (Horizontal) at $(0,0)$ but there is No max or min.

Example: The graph of function $f(x) = 3x^4 - 16x^3 + 18x^2 \quad -1 \leq x \leq 4$ Looks as the following, find all the local, and global **Max** and **Min Points**?



$(-1, 37)$ global Max (but not local Max because it is at the end point)

$(1, 5)$ Local Max

$(3, -27)$ global Min & Local Min

$(4, 32)$ Neither Local Max Nor global Max

There are Two important definitions that are usually used together when solving problems:

Definition: A critical number of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ does NOT exist

Definition: If f has local max or local min at point c, then c is a critical number of f (**Very importan Note:** the converse of this is NOT correct in general, meaning, if $f'(c)=0$ for a function, this does Not mean that there exist a local Min or a Local Max. $f(x)=x^3 \rightarrow f'(x)=3x^2 \Rightarrow \underbrace{f'(0)=0}$

But There is No Max or Min (local)

Example: find the critical numbers of $f(x)=x^{\frac{3}{5}}(4-x)$?

Solution: $f'(x)=x^{3/5}(-1)+(4-x)\left(\frac{3}{5}x^{-\frac{2}{5}}\right)=-x^{3/5}+\dfrac{3(4-x)}{5x^{2/5}}=\dfrac{12-8x}{5x^{2/5}}$

$f'(x)=0$ if $12-8x=0 \Rightarrow x=\frac{3}{2}$

Also $f'(x)$ does Not exist when $x=0$

} Therefore, There are Two critical numbers for This function: $x=0$ & $x=\frac{3}{2}$

Procedure to find <u>Absolute Max.</u> and <u>Absolute Min.</u> values of a continuous function f on a closed interval $[a,b]$:

1) Find the values of f at the critical numbers of f in (a,b)
2) Find the values of f at the endpoints of the interval.
3) The largest of the values of step 1 and 2 is the absolute maximum value, The smallest of these values is the absolute minimum value.

Example: find the absolute max and min values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \leq x \leq 4$$

Solution: step 1: $f'(x) = 0 \Rightarrow$ (since the $f'(x)$ exists) $\Rightarrow 3x^2 - 6x = 0$

$\Rightarrow x = 0,\ x = 2$ (both lies in the interval)

$\Rightarrow f(0) = 1,\quad f(2) = -3$

step 2: we find the values of f at the endpoints of the interval

$$f\left(-\frac{1}{2}\right) = \frac{1}{8} \qquad f(4) = 17$$

step 3: Compare the values:

$$\begin{cases} f(4) = 17 \longrightarrow \text{Absolute Max value} \\ f(2) = -3 \longrightarrow \text{Absolute Min value} \end{cases}$$

HW Example: find the absolute Max and absolute min of function f:

$$f(t) = 2\cos t + \sin 2t, \quad \left[0, \frac{\pi}{2}\right]$$

step 1: $f'(t) = 0 \Rightarrow -2\sin t + 2\cos 2t = 0$

$\Rightarrow \cos 2t = \sin t \Rightarrow \cos^2 t - \sin^2 t = \sin t$

$\Rightarrow 1 - \sin^2 t - \sin^2 t = \sin t \Rightarrow 2\sin^2 t + \sin t - 1 = 0$

$\Rightarrow \sin t = \pm \dfrac{-1 \pm \sqrt{1+8}}{4} = \dfrac{-1 \pm 3}{4}$ → -1, $\frac{1}{2}$

$\sin t = -1 \Rightarrow t = 2k\pi + \frac{3\pi}{2}$ outside $\left[0, \frac{\pi}{2}\right]$

$\sin t = \frac{1}{2} \Rightarrow t = 2k\pi + \frac{\pi}{6}$ inside $\left[0, \frac{\pi}{2}\right]$

$f\left(\frac{\pi}{6}\right) = 2 \times \cos\frac{\pi}{6} + \sin\frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3}{2}\sqrt{3}$

step 2: $f(0) = 2\cos 0 + \sin 2\times 0 = 2\times 1 + 0 = 2$

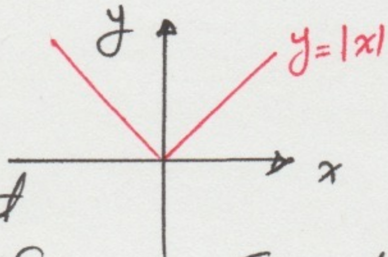
$f\left(\frac{\pi}{2}\right) = 2\cos\frac{\pi}{2} + \sin \pi = 0 - 0 = 0$

step 3: $f\left(\frac{\pi}{6}\right) = \frac{3}{2}\sqrt{3}$ Absolute Maximum

$f\left(\frac{\pi}{2}\right) = 0$ Absolute Minimum

An Application of Fermat's Theorem:

Example: The function $f(x)=|x|$ has a local & absolute min. value at $\underline{0}$, but that value can not be found using $f'(x)=0$ because as we have shown previously, $f(x)=|x|$ at $x=0$ is <u>NOT</u> differentiable or $f'(0)$ does not exist.



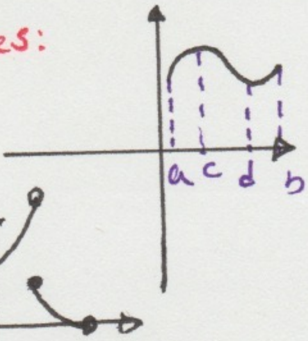
This example and the one for $f(x)=x^3$ shows that you have to be very careful using Fermat's Theorem.

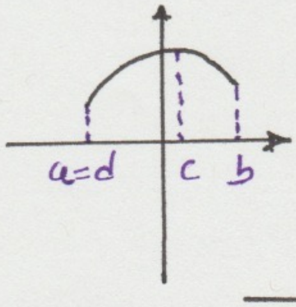
* if $f'(c)=0$, this does NOT mean we necessarily have a max or min at point c
* if we have a local Max or Min at point c, this does Not mean, $f'(c)$ is necessarily zero (it might Not exist)

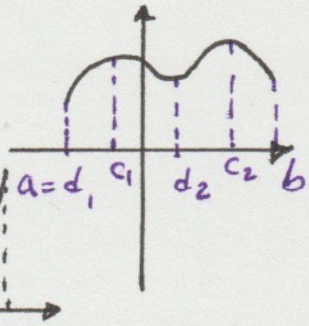
The Extreme Value Theorem:

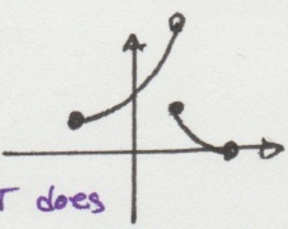
If f is continuous on a closed interval $[a,b]$, then f attains an absolute Max value, $f(c)$, and an absolute min. value $f(d)$ at some numbers c, and d in the interval $[a,b]$

Examples:

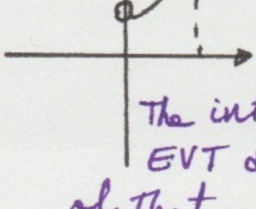








EVT does NOT work because the function is Not continuous.



The interval is Not closed and EVT does Not apply because of that $\Rightarrow$ No absolute Min or Max