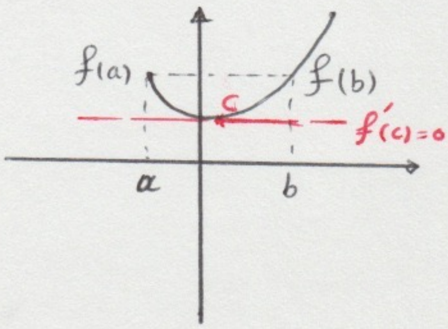
The Mean Value Theorem Section 4.2

Roll's Theorem: let f be a function that satisfies the following three conditions

1- f is continuous on the closed interval $[a,b]$

2- f is differentiable on the open interval (a,b)

3- $f(a) = f(b)$



Then there is a number c in (a,b) such that $f'(c) = 0$.

Example: prove that the equation $x^3 + x - 1 = 0$ has exactly one real root?

Solution: First we use **IVT** to show that the root exist:

$f(0) = 0 + 0 - 1 = -1 < 0$

$f(1) = 1 + 1 - 1 = 1 > 0$

$\Rightarrow$ According to IVT there is a number c $0 < c < 1$ such that $f(a) < f(c) < f(b)$

$f(a) = f(0) = -1$

$f(b) = f(1) = 1$

$\Rightarrow f(c) = 0$

Therefore, $x^3 + x - 1 = 0$ has a root in the interval $(0,1)$

To show that there exist only one root, we use Roll's Theorem and also argue by contraction.

suppose that the equation had Two roots in the interval, a, b.

$\Rightarrow f(a) = 0 = f(b)$

we know that f is a polynomial so it is Continuous on $[a,b]$ and it's differentiable on (a,b), since $f(a) = f(b)$, The Rolls Theorem holds.

$\Rightarrow$ we have a c such that $f'(c) = 0$ in the interval $[a,b]$

$f'(x) = 3x^2 + 1 \geq 1$ so $f'(x) \neq 0$ which means we can Not have $f(a) = f(b) = 0$

The Mean Value Theorem section 4.2

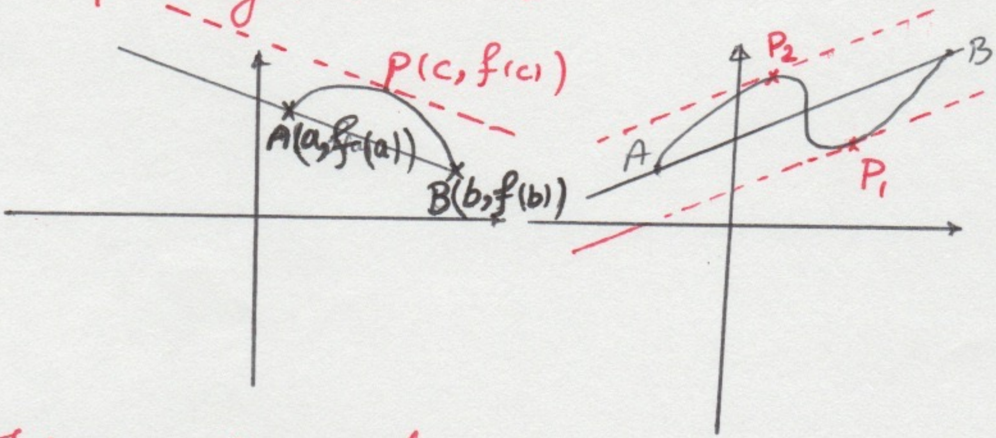
The mean Value Theorem states: Let f be a function that satisfies the following conditions:

1- f is continuous on the closed interval $[a,b]$

2- f is differentiable on the open interval (a,b)

then there is a number c in (a,b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Graphically This means



This Very innocent looking Theorem is extremely important and is used in many branches of Mathematics to solve Problems:

Example: for function $f(x) = x^3 - x$ in the interval $[\overset{a}{0}, \overset{b}{2}]$, Prove there is a point that tangent line to f at that point is parallel to the line that connect $(a, f(a))$ to $(b, f(b))$?

$x^3 - x$ is a polynomial so it is Continuous on $[0,2]$ and it is differentiable on $(0, 2)$ $\Rightarrow$ according to mean value Theorem there exist a $\underline{c}$ such That $f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{6-0}{2-0}$

$\Rightarrow f'(c) = 3x^2 - 1 = 3c^2 - 1 \Rightarrow 3c^2 - 1 = 3 \Rightarrow c = \pm\dfrac{2}{\sqrt{3}} \xrightarrow{\text{Acceptable}} c = \dfrac{2}{\sqrt{3}}$

Example: Suppose That $f(0)=-3$ and $f'(x) \leq 5$ for all values of x. How Large can $f(2)$ possibly be?

Solution: we are given That f is differentiable (and Therefore Continuous) everywhere. we Can apply The Mean Value Theorem in the interval $[0,2]$ $\Rightarrow$ we have

$$\frac{f(2)-f(0)}{2-0} = f'(c) \Rightarrow f(2) = 2f'(c) + \underbrace{f(0)}_{-3}$$

$$\left.\begin{array}{l} f(2) = -3 + 2f'(c) \\ f'(x) \leq 5 \end{array}\right\} \Rightarrow f(2) \leq -3 + 2(5)$$

$$\Rightarrow \boxed{f(2) \leq 7}$$

Example (HW set p:17): show That the equation $x^3+e^x=0$ has exactly one real root?

Solution: first we use Intermediate Value Theorem to show That There exist a root in $(-1,0)$ $\rightarrow f(x) = x^3+e^x$

$$\left.\begin{array}{l} f(-1) = -1 + e^{-1} = \frac{1}{e} - 1 < 0 \\ f(0) = 0 + e^0 = 1 > 0 \end{array}\right\}$$

According to IVT there exist a number c, $-1<c<0$ that $f(c)=0$.

Now we want to show That this is the only Solution That exist suppose That we have Two roots like a, and b between $(-1,0)$

$\Rightarrow f(a) = f(b) = 0$ $\quad x^3+e^x$ is Continuous and differentiable so we apply Roll's Theorem $\rightarrow$ There should be c such That $f'(c)=0$

$f'(x) = e^x + 3x^2 > 0 \Rightarrow$ we can Not have $f(a)=f(b)=0$