What Does f' say about f:

So far, we have seen that derivatives are closely related to rate of change of a function with respect to its independent variable

$\frac{df}{dt} \longrightarrow$ rate of change of f w/ respect to t

$\frac{dy}{dx} \longrightarrow$ rate of change of y wrt x

This means when rate of change is positive (Derivative) then the function is increasing

Also, when the rate of change is negative (Derivative) the the function is decreasing

> a) if $f'(x)>0$ on an interval, then f is increasing on that interval.
>
> b) if $f'(x)<0$ on an interval, then f is decreasing on that interval.

(The proof is very simple, you can find it in Page 290 of your text book. Mean Value Theorem is used for the proof.)

Example: find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing?

First we take the derivative: $f'(x) = 12x^3 - 12x^2 - 24x$

Now we want to know the sign of $f'(x) = 12x^3 - 12x - 24x = 12x(x-2)(x+1)$

We can now easily form the following table

	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x<-1$	$-$	$-$	$-$	$-$	Decreasing
$-1<x<0$	$-$	$-$	$+$	$+$	Increasing
$0<x<2$	$+$	$-$	$+$	$-$	Decreasing
$x>2$	$+$	$+$	$+$	$+$	Increasing

The first derivative test Introduction:

From previous lectures you remember that if f has a local maximum or minimum at c, then c must be a critical number of f, but Not every critical number gives rise to a maximum or a minimum.

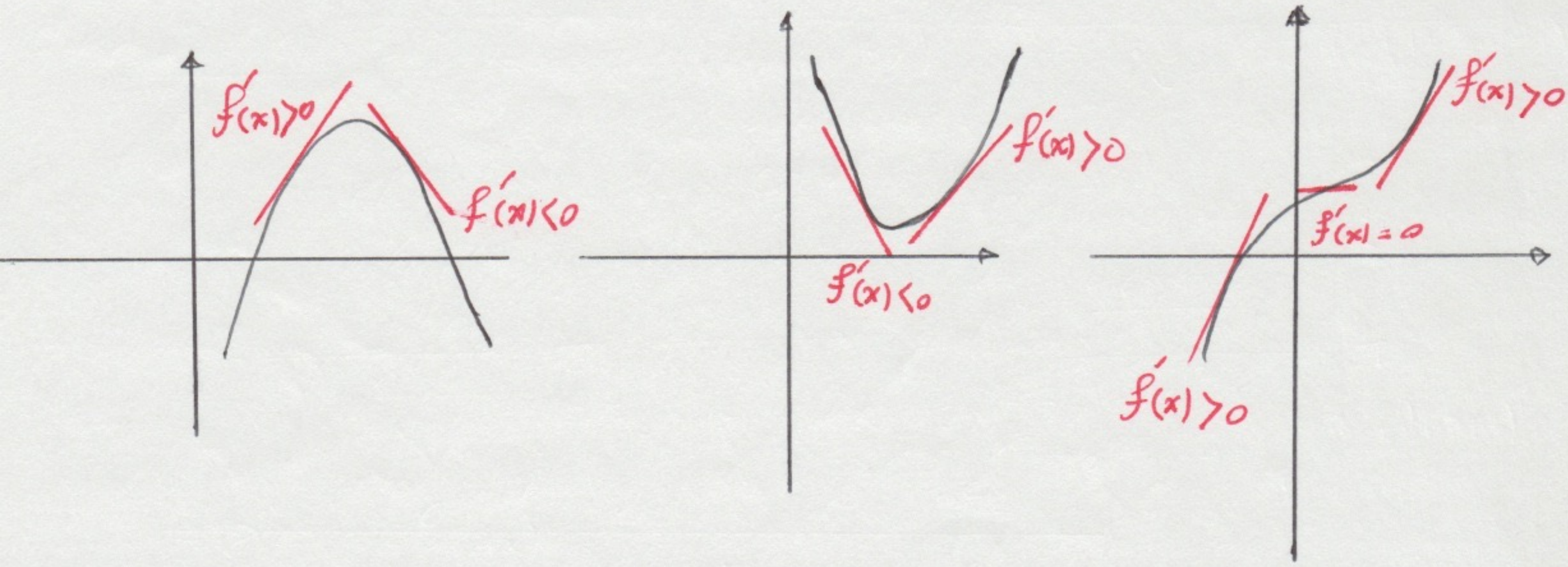
Now, we want to know what critical numbers give rise to local maximum or local minimum, and which ones does not give rise to locan max. or min. at all (remember the example of $y=x^3$).

The first derivative test:

a) if f' changes from positive to negative at c, then f has a local Maximum at c.

b) if f' changes from negative to positive at c, then f has a local Minimum at c.

c) If f' does Not change sign at c (positive at both sides of c or negative on both sides of c) then f has no local max. or min at c.



Example: find the local Min, and local Max values of the function f in Example 1/ (previous example)?

Solution: from the table we know that the critical points are

$x = -1$

$x = 0$

$x = 2$

$f'(x)$ changes from negative to positive at $\underline{-1}$: Local Minimum

$f'(x)$ changes from positive to negative at $\underline{0}$: Local Maximum

$f'(x)$ changes from Negative to positive at $\underline{2}$: Local Minimum

Example: Find the local maximum and local Minimum values of the function

$$g(x) = x + 2\sin x \quad 0 \leq x \leq 2\pi$$

Solution: $g'(x) = 1 + 2\cos x \Rightarrow g'(x) = 0$ when $\cos x = -\frac{1}{2} \Rightarrow \begin{cases} x = \frac{2\pi}{3} \\ x = \frac{4\pi}{3} \end{cases}$

because g is differentiable everywhere, the only critical numbers are $\frac{2\pi}{3}$ & $\frac{4\pi}{3}$

(we do not need to worry about the point that the derivative of g does not exist)

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < \frac{2\pi}{3}$	+	increasing
$\frac{2\pi}{3} < x < \frac{4\pi}{3}$	−	decreasing
$\frac{4\pi}{3} < x < 2\pi$	+	increasing

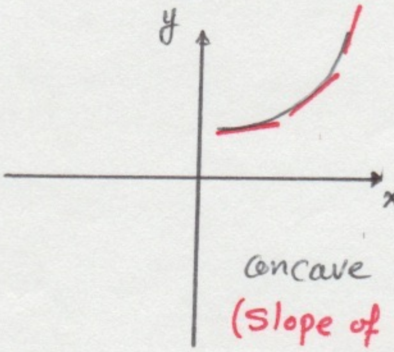
According to the first derivative test:

$x_1 = \frac{2\pi}{3}$ from positive to negative : Local Maximum : $g(\frac{2\pi}{3}) = \sqrt{3} + \frac{2\pi}{3}$

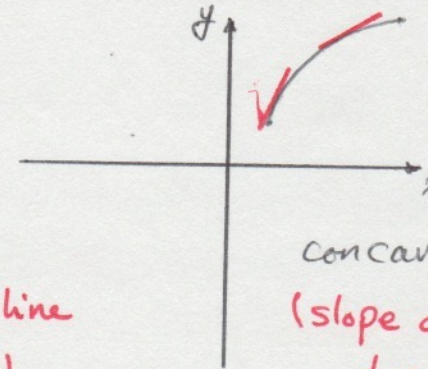
$x_2 = \frac{4\pi}{3}$ from negative to positive : Local Minimum : $g(\frac{4\pi}{3}) = -\sqrt{3} + \frac{4\pi}{3}$

What does f'' say about f:

Definition: If the graph of f lies above all of its tangents, on an interval I, Then, it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.



concave upward
(slope of tangent line is increasing)



concave downward.
(slope of tangent line is decreasing)

Concavity Test:

a) if $f''(x) > 0$ for all x in I, Then the graph of f is concave upward on I.

b) if $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition: A point P on curve $y = f(x)$ is called an inflection point if f is continuous there and curve changes from concave upward to concave downward or from concave downward to concave upward at P.

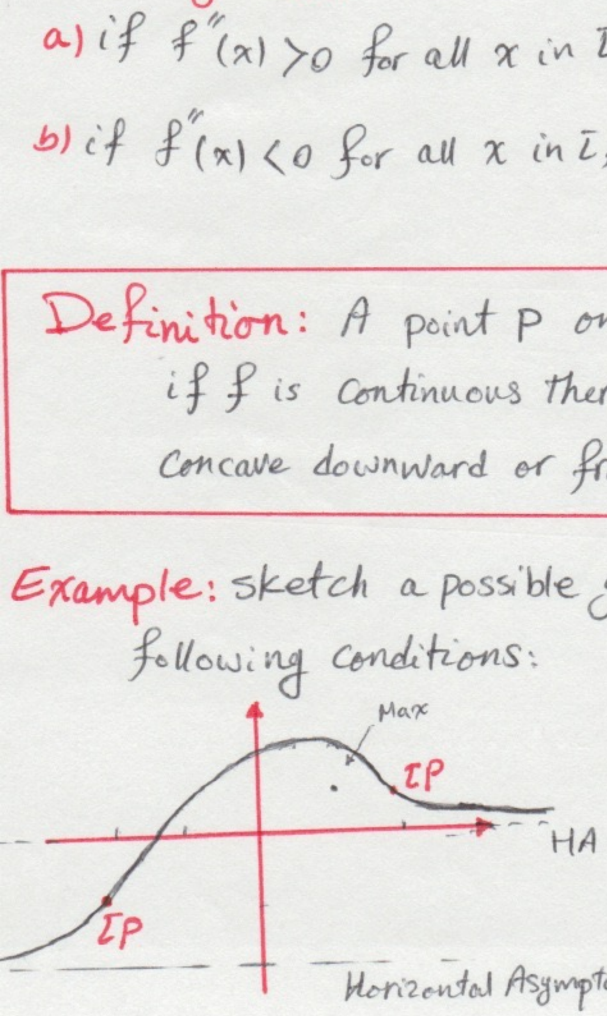
Example: sketch a possible graph of a function f that satisfies the following conditions:

$f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

$f''(x) > 0$ on $(-\infty, -2)$, and $(2, \infty)$

$f''(x) < 0$ on $(-2, 2)$

$\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$



The second derivative test: Suppose f'' is continuous near c.

a) If $f'(c)=0$ and $f''(c)>0$, Then f has a local minimum at c.

b) If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.

Example: Discuss the curve $y=x^4-4x^3$ wrt, Concavity, point of inflection, and local maxima and minima. Then use These information to sketch the curve?

Solution: $f'(x)=4x^3-12x^2=4x^2(x-3)$

$f''(x)=12x^2-24x=12x(x-2)$

Then we find the critical numbers of the function by setting $f'(x)=0$

$\Rightarrow x=0$, $x=3$ (critical point)

Since we want to use The second derivative test, we evaluate f'' at the critical points: $f''(0)=0$, $f''(3)=36>0$

$f'(3)=0$ and $f''(3)>0$, Then $f(3)=-27$ is a local minimum

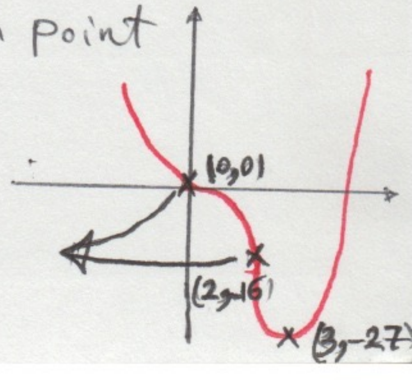
$f''(0)=0$ (so The second derivative test does not give us any information)

since $f'(x)<0$ for $x<0$ and $0<x<3$ Then the first derivative test tells us That There is no local maximum or minimum at $x=0$

Interval	$f''(x)=12x(x-2)$	Concavity
$(-\infty,0)$	+	upward
$(0,2)$	−	downward
$(2,8)$	+	upward

$(0,0) \rightarrow$ inflection point

$(+2,-16) \rightarrow$ inflection point



 How derivatives affect the shape of a graph section 4.3

Example: sketch the graph of function $f(x) = x^{\frac{2}{3}}(6-x)^{1/3}$?

Solution: $f'(x) = \dfrac{4-x}{x^{1/3}(6-x)^{2/3}}$, $f''(x) = \dfrac{-8}{x^{4/3}(6-x)^{5/3}}$

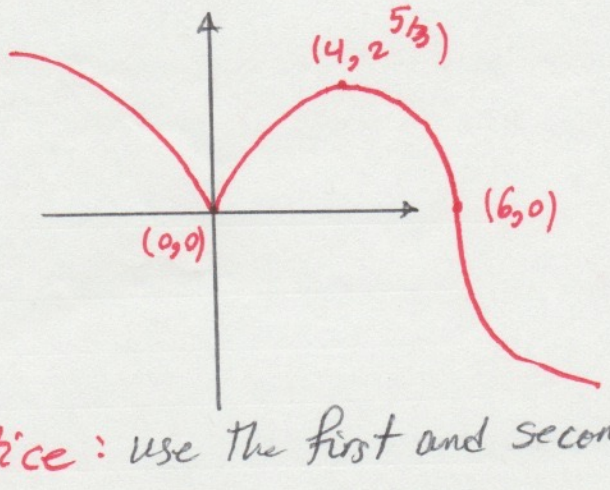
$f'(x)=0$ when $x=4$ and $f'(x)$ does not exist when $x=0$ & $x=6$

Interval	$4-x$	$x^{1/3}$	$(6-x)^{2/3}$	$f'(x)$	f
$x<0$	+	−	+	−	Decreasing $(-\infty, 0)$
$0<x<4$	+	+	+	+	Increasing $(0,4)$
$4<x<6$	−	+	+	−	Decreasing $(4,6)$
$x>6$	−	+	+	−	Increasing $(6,\infty)$

$\Rightarrow$ $x=0 \to$ local minimum } The sign of $f'(x)$ is changing
$x=4 \to$ Local maximum

Interval	$x^{4/3}$	$(6-x)^{5/3}$	$f''(x)$	f
$x<0$	+	+	−	concave downward
$0<x<6$	+	+	−	concave downward
$x>6$	+	−	+	concave upward

$\Rightarrow$ $x=6$ is an inflection Point of graph



$(0,0) \to$ Local minimum

$(6,0) \to$ inflection Point

$(4, 2^{5/3}) \to$ Maximum

Practice: use the first and second derivatives of $f(x) = e^{1/x}$, together w/ asymptotes to sketch the graph.