Indeterminate Forms and l'Hospital's Rule Section 4.4

Even Though the following function is not defined when $x=1$, we need to know How F behaves near 1. In other words, we would like to know:

$$F(x) = \frac{\ln x}{x-1} \qquad \lim_{x\to 1} \frac{\ln x}{x-1}$$

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \frac{0}{0}$$ (This is called an Indeterminate form of type $\frac{0}{0}$)

Another situation That might occur is:

$$\lim_{x\to \infty} \frac{\ln x}{x-1} = \frac{\infty}{\infty}$$ (This is called an Indeterminate form of type $\frac{\infty}{\infty}$)

In These Cases we can use L'Hospital's Rule

- L'Hospitals Rule: If we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \underbrace{\lim_{x\to a} \frac{f'(x)}{g'(x)}}_{\text{if this limit exist, of course!}}$$

Example: find $\lim_{x\to 1} \frac{\ln x}{x-1}$?

Solution $\lim_{x\to 1} \frac{\ln x}{x-1} = \frac{0}{0}$ because $\ln 1 = 0$ and $\begin{cases} x-1=0 \\ x=1 \end{cases}$

so we can use L'Hospital's Rule: $$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x\to 1} \frac{1}{x} = 1$$

Example: Calculate $\lim\limits_{x\to\infty} \frac{e^x}{x^2}$?

Solution: $\lim\limits_{x\to\infty} e^x = \infty$ and $\lim\limits_{x\to\infty} x^2 = \infty \Rightarrow \lim\limits_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$

Therefore, we can use L'Hospital's Rule.

$$\Rightarrow \lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x\to\infty} \frac{e^x}{2x} = \frac{\infty}{\infty} \text{ (Indeterminate type } \tfrac{\infty}{\infty})$$

Therefore, we can use L'Hospital's Rule Again:

$$\lim_{x\to\infty} \frac{e^x}{2x} = \frac{\infty}{\infty} \Rightarrow \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}(2x)} = \lim_{x\to\infty} \frac{e^x}{2} = \frac{e^\infty}{2} = \infty$$

Example: Calculate $\lim\limits_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$?

Solution:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \frac{\infty}{\infty} \Rightarrow \lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\sqrt[3]{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-\frac{2}{3}}}$$

$$= \lim_{x\to\infty} \frac{\frac{3}{x}}{x^{-\frac{2}{3}}} = \lim_{x\to\infty} \frac{3}{x^{1/3}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0$$

Example: find $\lim_{x \to 0} \frac{\tan x - x}{x^3}$?

Solution: $\lim_{x \to 0} \frac{\tan x - x}{x^3} = \frac{0}{0} \rightarrow$ L'Hospital's: $\lim_{x \to 0} \frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}(x^3)}$

$$= \lim_{x \to 0} \frac{(1+\tan^2 x - 1)}{3x^2} = \lim_{x \to 0} \frac{\tan^2 x}{3x^2} = \frac{0}{0} \Rightarrow \lim_{x \to 0} \frac{\tan^2 x}{3x^2} = \lim_{x \to 0} \frac{\frac{d}{dx}(\tan^2 x)}{\frac{d}{dx}(3x^2)}$$

$$\Rightarrow = \lim_{x \to 0} \frac{2\tan x\,(1+\tan^2 x)}{6x} = \lim_{x \to 0} \frac{\tan x \times 1}{3x} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{1+\tan^2 x}{3} = \frac{1}{3}$$

Example: find $\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$ (This is a trick question)

If you blindly try to use L'Hospital's Rule, it wont work because we donot have an indeterminate form $\frac{0}{0}$

$$\lim_{x \to \pi^-} \frac{\sin x}{1-\cos x} = \frac{\sin \pi}{1-(-1)} = \frac{0}{2} = 0$$

The take home message is, do Not try to use the L'Hospital's Rule on every limit problem that you see.

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Indeterminate products:

Sometimes, when we are calculating $\lim_{x \to a} [f(x)\, g(x)]$ we encounter another indeterminate form of type $0 \cdot \infty$, in these cases we can convert

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

to get the indeterminate form $\frac{0}{0}$ and $\frac{\infty}{\infty}$

and then use L'Hôspital's Rule.

Example: Evaluate $\lim_{x \to 0^+} x \ln x$?

$$\lim_{x \to 0^+} x = 0, \quad \lim_{x \to 0^+} \ln x = -\infty \Rightarrow \lim_{x \to 0^+} x \ln x = 0 \times -\infty$$

$$\Rightarrow \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \frac{-\infty}{\infty} \longrightarrow \text{we use L'Hospital's Rule}$$

$$\Rightarrow \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{-1/x^2} = \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} \frac{-x^2}{x} = \lim_{x \to 0^+} (-x) = 0$$

Note: We could also say $\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{x}{1/\ln x}$

Indeterminate Differences:

Another indeterminate form that sometimes we encounter is $\infty - \infty$

In these cases we try to convert the difference into a quotient (for example, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example: Compute $\lim_{x \to \frac{\pi}{2}^-} (\sec x - \tan x)$?

We notice that $\sec x \to \infty$ and $\tan x \to \infty$ when $x \to \frac{\pi}{2}^-$

Here we use common denominator:

$$\Rightarrow \lim_{x \to \frac{\pi}{2}^-} (\sec x - \tan x) = \lim_{x \to \frac{\pi}{2}^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \lim_{x \to \frac{\pi}{2}^-} \frac{1 - \sin x}{\cos x} = \frac{0}{0}$$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} = \lim_{x \to \frac{\pi}{2}^-} \frac{0 - \cos x}{-\sin x} = 0$$

Indeterminate Powers: Several indeterminate forms arise from the limit:

$$\lim_{x \to a} [f(x)]^{g(x)} \Rightarrow \begin{array}{l} 1)\ (0)^0 \\ 2)\ \text{Type } \infty^0 \\ 3)\ \text{Type } 1^\infty \end{array}$$

Each of these three cases can be treated by taking the natural Logarithm:

$$y = f(x)^{g(x)} \Rightarrow \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$y = f(x)^{g(x)} \Rightarrow y = e^{g(x) \ln f(x)}$$

Then we get to the indeterminate form $0 . \infty$ that we can solve.

Example: Calculate $\lim_{x \to 0^+} (1+\sin 4x)^{\cot x}$?

$$\lim_{x \to 0^+} (1+\sin 4x)^{\cot x} = 1^\infty \Rightarrow y = (1+\sin 4x)^{\cot x} \Rightarrow \ln y = \cot x \ln(1+\sin 4x)$$

$$\Rightarrow \lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \cot x \ln(1+\sin 4x) = 0 \times \infty$$

$$= \lim_{x \to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} = \frac{0}{0} \xrightarrow[\text{use l'Hospital's}]{\text{now we}} \lim_{x \to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{1+\tan^2 x} = 4$$

$$\Rightarrow \lim_{x \to 0^+} (1+\sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$

Example: find $\lim_{x \to 0^+} x^x$