General Guidelines for sketching a curve:

The following checklist is intended as a guide to sketching a curve $y=f(x)$ by hand. Not every item is relevant to every function (for instance, a given curve might not have an asymptote or posses symmetry) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

1) **Domain**: the set of values of x, which $f(x)$ is defined
2) **Intercepts**: a) y-intercept $\to f(0)$ b) x-intercepts $\to y=0$
3) **Symmetry**: $f(x)=f(-x)$: symmetric about y $f(-x)=-f(x)$: symmetric about origin
4) **Asymptotes**: Horizontal, vertical, and slant $f(x+P)=f(x)$: periodic function
5) **Interval of increase or decrease**: Compute $f'(x)$ and use I/D test (Table $f'(x)$)
6) **Local Max and Min Values**: find critical points ($f'(x)=0$ or $f'(x)$ does Not exist) Then use first derivative test or second derivative test to find local Max. and local Min Points.
7) **Concavity and points of inflection**: Compute $f''(x)$ and use the concavity Test. find the inflection points (when the direction of concavity changes)
8) **Sketch the Curve**:

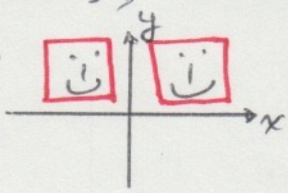
For More details, refer to Page 315, and 316 of your book

Example: use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2-1}$

Solution: 1: $D = \{x \mid x \neq \pm 1\}$

2: $\begin{cases} f(0)=0 \\ y=0 \Rightarrow x=0 \end{cases} \Rightarrow (0,0)$ is an intercept point (intercepts both x-axis and y-axis at the same point)

3: $f(-x)=f(x)$: Symmetric about y-axis means whatever shape we have on the righthand side of y-axis, we have the something on the left hand side.



4: $\lim\limits_{x \to \pm\infty} \dfrac{2x^2}{x^2-1} = 2 \Rightarrow \boxed{y=2}$ (Horizontal Asymptote)

$\lim\limits_{x \to 1^+} \dfrac{2x^2}{x^2-1} = +\infty$, $\lim\limits_{x \to 1^-} \dfrac{2x^2}{x^2-1} = -\infty$, $\lim\limits_{x \to -1^-} \dfrac{2x^2}{x^2-1} = +\infty$

$\lim\limits_{x \to -1^+} \dfrac{2x^2}{x^2-1} = -\infty$ $\boxed{x = \pm 1}$ (vertical Asymptotes)

5: $f'(x) = \dfrac{4x(x^2-1)-(2x)(2x^2)}{(x^2-1)^2} = \dfrac{-4x}{(x^2-1)^2}$ $x>0 \to f'(x)<0$
$x<0 \to f'(x)>0$

$\Rightarrow$ f from $(-\infty,-1)$ and $(-1,0)$ is increasing
f from $(0,1)$ and $(1,+\infty)$ is decreasing

$f'(0)=0$ critical Point
$(0,0)$ Local Max

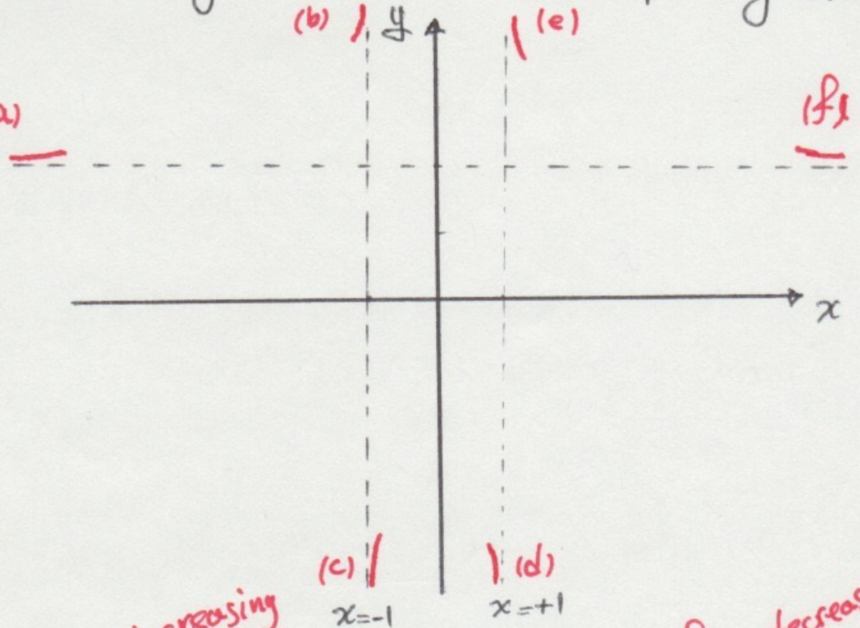
6: $f''(x) = \dfrac{12x^2+4}{(x^2-1)^3}$ (Do this part yourself to get some practice)

$12x^2+4$ is always positive, therefore, the sign of $f''(x)$ depends of the sign of (x^2-1). $x^2-1<0 \Rightarrow -1<x<1 \to f''(x)<0$

$(x>1)$ or $(x<-1) \to f''(x)>0$

$(-1,1)$ Concave downward
$(-\infty,-1)$ and $(1,+\infty)$ Concave upward

Now we go ahead and start putting everything together.



$$y = \frac{2x^2}{x^2-1}$$

a: first we plot all the asymptotes

b: when $x \to -1^-$ (values smaller than -1, $f(x)$ approaches $+\infty$)

c: when $x \to -1^+$ (Values larger than -1, $f(x)$ approaches $-\infty$)

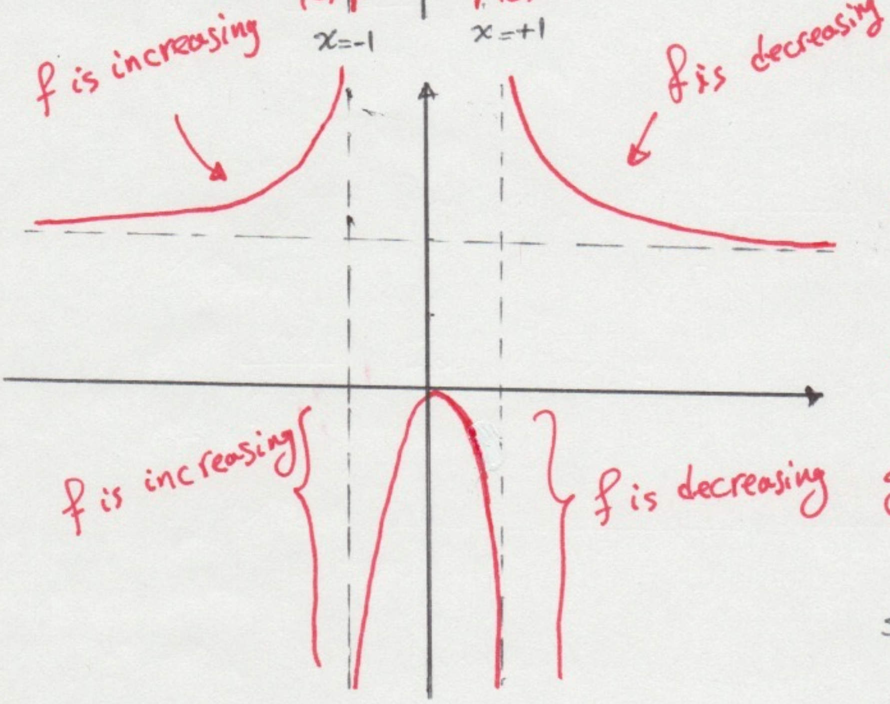
d: when $x \to +1^-$ (Values smaller than $+1$, $f(x)$ approaches $-\infty$

e: when $x \to +1^+$ (values larger than $+1$) $f(x) \to +\infty$

f: Another Horizontal asymptote point $x \to +\infty$, $y \to 2$

g: from $(-\infty, -1)$ and $(-1, 0)$ the function is increasing so we connect (a) to (b) and (c) to the origin.

from $(0, 1)$ and $(1, +\infty)$, f is decreasing so we connect origin to (d) and (e) to (f).



As you can see the curve is symmetric about y-axis

As you can see, f is concave upward from $(-\infty, -1)$, and $(1, +\infty)$

Also f is concave downward from $(-1, 1)$

Example: sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$

Sol.: Domain $= (-1, \infty)$

the x intercept and y intercept are both 0

There is No symmetry

$$\lim_{x \to -1^+} \frac{x^2}{\sqrt{x+1}} = +\infty \Rightarrow x = -1 \text{ (vertical Asymptote)}$$

There is no Horizontal asymptote

$$f'(x) = \frac{2x\sqrt{x+1} - x^2 \cdot \frac{1}{2\sqrt{x+1}}}{x+1} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

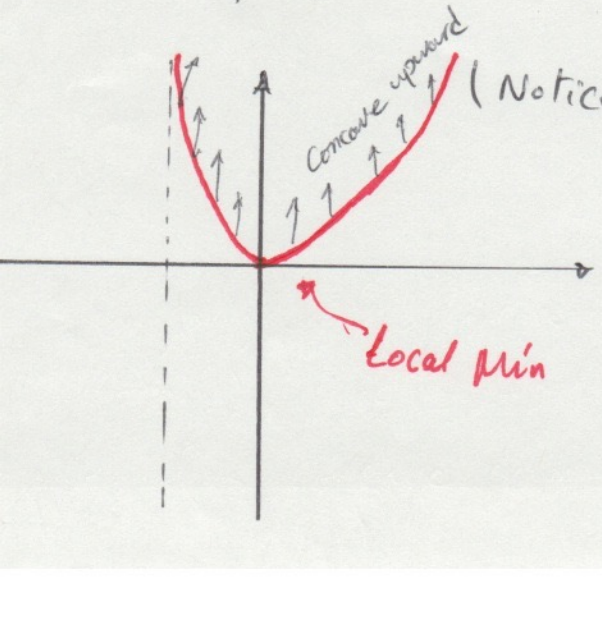
$f'(x) = 0 \Rightarrow x = 0$ and $x = \frac{-4}{3}$ (outside the domain)

$-1 < x < 0 \rightarrow f'(x) < 0 \rightarrow f$ is decreasing

$x > 0 \rightarrow f'(x) > 0 \rightarrow f$ is increasing

$f'(0) = 0$, f' changes sign from negative to positive at $x = 0$, Then $f(0) = 0$ is a Local minimum (By The first derivative Test)

$f''(x) = \dfrac{3x^2 + 8x + 8}{4(x+1)^{5/2}}$ (Notice that the denominator is always positive) Why?



(Notice that the numerator is always positive) Why? (Hint: it is a quadratic form with $b^2 - 4ac < 0$)

$\Rightarrow f''(x) > 0$ So The Curve is always Concave upward.

Example: sketch the graph of $f(x) = xe^{x}$

Solution:

1) the domain is $\mathbb{R}$

2) x and y intercepts are both zero (0)

3) There is No symmetry

4) $\lim\limits_{x \to +\infty} xe^{x} = \infty$ No Horizontal asymptote whe $x \to +\infty$

$\lim\limits_{x \to -\infty} xe^{x} = 0 \times \infty$ (Indeterminate form $0 \times \infty$) $\Rightarrow \lim\limits_{x \to -\infty} xe^{x} = \lim\limits_{x \to -\infty} \dfrac{x}{e^{-x}} = \dfrac{\infty}{\infty}$

So we use L'Hospital's Rule $\Rightarrow = \lim\limits_{x \to -\infty} \dfrac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^{-x})} = \lim\limits_{x \to -\infty} \dfrac{1}{-e^{-x}} = 0$

$\Rightarrow$ $x = 0$ is a horizontal asymptote

5) $f'(x) = 1 \times e^{x} + xe^{x} = e^{x}(1+x)$

e^{x} is always positive therefore, if $x+1 > 0$ Then $f'(x) > 0$

if $x+1 < 0$ Then $f'(x) < 0$

$x > -1 \to f$ is increasing

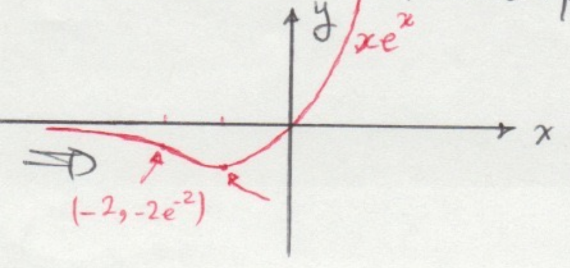
$x < -1$ f is decreasing

6) $f'(-1) = 0$ and f' changes from negative to positive, then $x = -1$ is a ((local and absolute) minimum.

7) $f''(x) = (x+2)e^{x}$ $\Rightarrow$ $x > -2 \to f''(x) > 0$ Concave upward

$x < -2 \to f''(x) < 0$ Concave downward

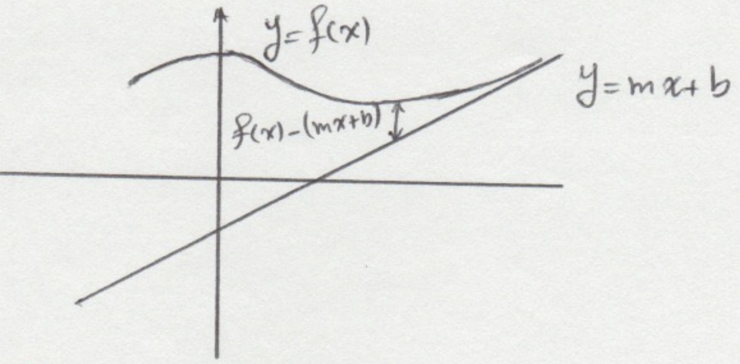
we have an inflection point at $(-2, f(-2))$ or $(-2, -2e^{-2})$

$\Rightarrow$ 

Slant Asymptotes:

Some curves have asymptotes that are oblique, that is, neither Horizontal nor vertical.

If $\lim\limits_{x\to\infty}\left[f(x)-(mx+b)\right]=0$ where $m\neq 0$, then the line $y=mx+b$ is called a slant asymptote. (because the vertical distance between $y=f(x)$ and the line $y=mx+b$ approaches to $\underline{0}$ as x approaches ∞)



Example: Sketch the graph of $f(x)=\dfrac{x^3}{x^2+1}$?

Domain $=(-\infty,+\infty)$, the x- and y- intercepts are both 0.

$f(-x)=-f(x)\longrightarrow$ then f is odd $\longrightarrow$ symmetry about origin

Since x^2+1 is never zero, we do NOT have vertical asymptotes

$\lim\limits_{x\to\pm\infty}\dfrac{x^3}{x^2+1}=\pm\infty$ (There is No Horizontal asymptote either)

But $f(x)=\dfrac{x^3}{x^2+1}=x-\dfrac{x}{x^2+1}\Rightarrow f(x)-x=\dfrac{-x}{x^2+1}\Rightarrow\lim\limits_{x\to\pm\infty}f(x)-x=\lim\limits_{x\to\pm\infty}\dfrac{-x}{x^2+1}=0$

$\Rightarrow$ **$y=x$ is a slant asymptote.**

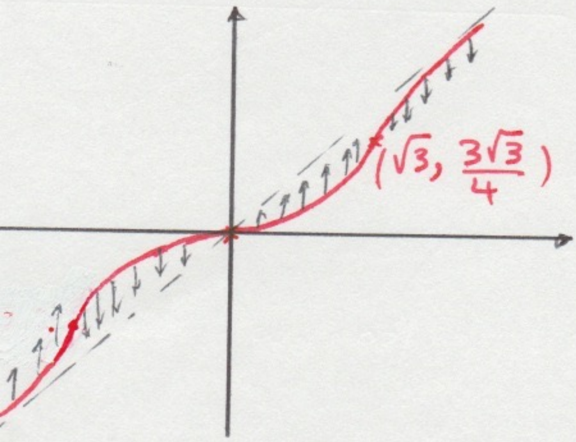
$f'(x)=\dfrac{x^2(x^2+3)}{(x^2+1)^2}\Rightarrow f'(x)>0$ There is $\underline{\text{NO}}$ Local Max or Local Min.

$$f''(x) = \frac{2x(3-x^2)}{(x^2+1)^3}$$

$f''(x) = 0 \Rightarrow x = 0$

$x = \pm\sqrt{3}$

Interval	x	$3-x^2$	$(x^2+1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD



CU: Concave upward

CD: Concave Downward