finding extreme values of a function has practical applications in many areas of life and science.

1) A business person wants to minimize costs and maximize Profits.

2) A Traveler wants to minimize the transportation time.

This chapter deals with such optimization problems.

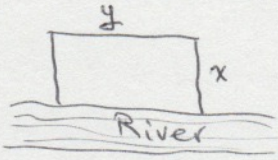
Note: In solving practical problems, often understanding the problem and converting the word problem into a mathematical optimization problem is often the greatest challenge.

The steps in solving optimization problems include:

1) understand the problem
2) Draw a diagram
3) Introduce Notation
4) when you are trying to minimize or maximize a function, express that function in terms of other variables or **symbols**.
5) If the function includes (expressed) in terms of more than one variable, use the given info in the problem to eliminate extra parameters and define the function in terms of <u>only one</u> variable
6) use the method that was explained in chapter 4.1 and 4.3 to find the absolute Min or absolute Max of the function.

Example: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution:



* The problem is asking for the largest area.
* we need to find x and y such that area is the largest

given: 2400 ft of fencing
No need to fence along the river

$\Rightarrow$ $2x + y = 2400$ ft (given by the problem) $\Rightarrow y = 2400 - 2x$

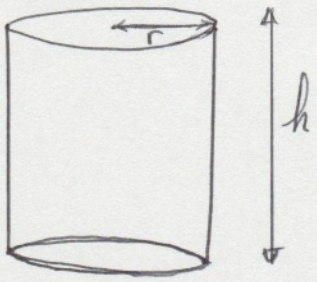
Area $= A = xy \Rightarrow A(x) = x(2400 - 2x) = 2400x - 2x^2$

Note that if $0 \le x \le 1200$ then $A \ge 0$ therefore the domain of function A is $0 \le x \le 1200$

we set $A'(x) = 0 \Rightarrow 2400 - 4x = 0 \Rightarrow x = 600$

$A''(x) = -4$, therefore according to the second derivative test we have a Max at $x = 600$ ft $\Rightarrow y = 1200$ ft (these give the max area of rectangular shape)

Example: A cylindrical Can is to be made to hold 1 Lit of oil. Find the dimensions that will minimize the cost of the metal to manufacture the Can?



$$A = \underbrace{2\pi r^2}_{\substack{\text{top}\\ \text{and}\\ \text{buttom}}} + \underbrace{2\pi r h}_{\text{Sidings}} \quad \text{(Area)}$$

$$V = \pi r^2 h \quad \text{(Volume)} = 1000 \text{ cm}^3$$

$$\Rightarrow h = \frac{1000}{\pi r^2}$$

$$\Rightarrow A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

$$\Rightarrow A(r) = 2\pi r^2 + \frac{2000}{r} \quad (r > 0)$$

Now to find the critical numbers of A, we take its derivative and set it to zero.

$$\Rightarrow A'(r) = 4\pi r - \frac{2000}{r^2} = 0 \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$$

We Can observe that (try this) for $r < \sqrt[3]{\frac{500}{\pi}}$, $A'(r) < 0$

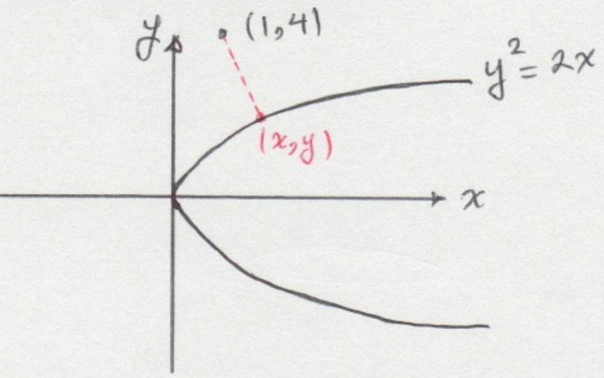
for $r > \sqrt[3]{\frac{500}{\pi}}$, $A'(r) > 0$

This means that $r = \sqrt[3]{\frac{500}{\pi}}$ is the absolute minimum. Why?

$$h = \frac{1000}{\pi r^2} \Rightarrow h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

So to maximize the volume, $r = \sqrt[3]{\frac{500}{\pi}}$ and $h = 2r$

Example: find the point on the parabola $y^2 = 2x$, That is closest to The point $(1,4)$?



Solution: the distance between the point $(1,4)$ and (x,y) is $d = \sqrt{(x-1)^2 + (y-4)^2}$

we know that (x,y) lies on the parabola, so $x = \frac{1}{2}y^2$

$$\Rightarrow d = \sqrt{\left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2}$$

We need to convince ourselves that whatever the value for x and y minimizes d, they will also minimize d^2, therefore:

(it is easier to work with d^2)

$$f(y) = d^2 = \left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

$$\Rightarrow f'(y) = 2(y)\left(\frac{1}{2}y^2 - 1\right) + 2(y-4) = y^3 - 8$$

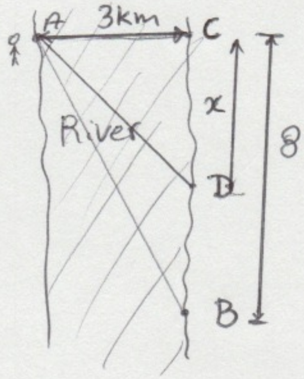
$$\Rightarrow f'(y) = 0 \Rightarrow y = 2 \text{ and } x = 2$$

$$\left.\begin{array}{l} \text{if } y > 2 \Rightarrow f'(y) > 0 \\ \text{if } y < 2 \Rightarrow f'(y) < 0 \end{array}\right\} \Rightarrow$$

According to the first derivative test, The absolute extreme value occurs at $y=2$ and is a minimum.

$$\Rightarrow \begin{array}{l} x = 2 \\ y = 2 \end{array} \text{ minimizes } f(y) \text{ and.}$$

Example: A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to the point C and then run to B, or he could row directly to B, or He could row to some point D between C and B, then run to B. If He can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible?



Solution: Lets say this man needs to row to point D and then run from D to B. We do <u>NOT</u> know where D is located but the distance from C to D, we assume is x. $x=0$ means he rows to C and then run to B and $x=8$ means that He rows directly to B.

$CB = 8$ km, $CD = x \Rightarrow BD = 8 - x$ (the distance that he needs to run)

Pythagorean: $AC = 3$, $CD = x \Rightarrow AD = \sqrt{x^2+9}$ (The distance that he needs to row)

$\Rightarrow$ Velocity (rate) $= \dfrac{\text{Distance}}{\text{time}} \Rightarrow \text{time} = \dfrac{\text{Distance}}{\text{rate}}$ (we are interested in minimizing the time)

$\Rightarrow$ time that he needs to run $= \dfrac{8-x}{8}$

time that he needs to row $= \dfrac{\sqrt{x^2+9}}{6}$ $\Rightarrow$ Total time $T(x) = \dfrac{\sqrt{x^2+9}}{6} + \dfrac{8-x}{8}$

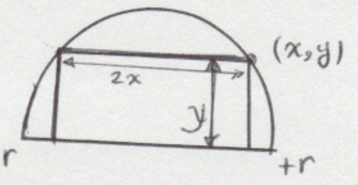
$$\Rightarrow T'(x) = \frac{2x}{6 \times 2\sqrt{x^2+9}} - \frac{1}{8} = 0 \Rightarrow 8x = 6\sqrt{x^2+9} \Rightarrow 16x^2 = 9(x^2+9)$$

$$\Rightarrow x = \frac{9}{\sqrt{7}}$$

$x = \dfrac{9}{\sqrt{7}}$ is the critical number.

By the first derivative test, we can see that $x = \dfrac{9}{\sqrt{7}}$ is the point that $T(x)$ is minimized.

Example: find the area of the largest rectangle that can be inscribed in a semicircle of radius r?



lets say we take the upper half of semi-circle $x^2+y^2=r^2$

we pick a point like (x,y) that will be on the semicircle as well as a corner of rectangle.

$A = 2xy$ Area of rectangle

$y = \sqrt{r^2-x^2} \Rightarrow A = 2x\sqrt{r^2-x^2}$

we want to find the critical point of $A(x) \Rightarrow A'(x) = 0$

$$A'(x) = 2\sqrt{r^2-x^2} + \frac{2x\cdot(-2x)}{2\sqrt{r^2-x^2}} = 2\sqrt{r^2-x^2} - \frac{4x^2}{\sqrt{r^2-x^2}} = \frac{2(r^2-2x^2)}{\sqrt{r^2-x^2}}$$

$\Rightarrow r^2 = 2x^2 \Rightarrow x = \frac{r}{\sqrt{2}}$

$$\left.\begin{array}{l} \text{if } x > \frac{r}{\sqrt{2}} \rightarrow A'(x) < 0 \\ x < \frac{r}{\sqrt{2}} \rightarrow A'(x) > 0 \end{array}\right\}$$

Therefore according to the first derivative test $x = \frac{r}{\sqrt{2}}$ is the maximum point of function $A(x)$