- So far, every problem that we have seen, we have had a function that we wanted to take its derivative.

Now, suppose that we have the derivative of a function and we want to obtain the original function that we have its derivative in hand.

For example, we know that we are asked to find the function that its derivative is $2x$. Since $\frac{d}{dx}(x^2) = 2x$, we know that the original function should have been in the form of $x^2 + c$ (c is a constant). So when we take the derivative, x^2 becomes $2x$ and the derivative of c, which is a constant goes to zero.

Definition: A function F is called antiderivative of f on an interval I if

$$F'(x) = f(x) \text{ for all } x \text{ in } I$$

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$

where C is an arbitrary constant.

Example: Find the most general antiderivative of each of the following functions.

$f(x) = \sin x$ $\quad f(x) = \frac{1}{x}$ $\quad f(x) = x^n$

if $F(x) = -\cos x \Rightarrow F'(x) = \frac{d}{dx}(-\cos x) = \sin x \Rightarrow$ the antiderivative of $\sin x$ becomes $-\cos x$, according to the Theorem the most general form becomes

$F(x) = -\cos x + C$

If $F(x) = \ln|x| \Rightarrow F'(x) = \frac{1}{x} \Rightarrow F(x) = \begin{cases} \ln x + C & x > 0 \\ \ln(-x) + c & x < 0 \end{cases}$

remember $\ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$

if we have $F(x) = \frac{1}{n+1} x^{n+1} \Rightarrow F'(x) = (n+1) \frac{1}{n+1} x^{n+1-1} = x^n$

$\Rightarrow$ if $f(x) = x^n \Rightarrow F(x) = \frac{x^{n+1}}{n+1} + C$

Table of antiderivatives: (very important)

Function	Particular Antiderivative (w/o Constant c)
$\alpha f(x)$	$aF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln \lvert x \rvert$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\tan x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$

Example: Find all functions g such that:

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x} \quad ?$$

Solution: First, lets rewrite the function g' and simplify it.

$$\Rightarrow g'(x) = 4\sin x + \frac{2x^5}{x} - \frac{x^{1/2}}{x} = 4\sin x + 2x^4 - x^{-1/2}$$

using table of antiderivatives we have:

$$G(x) = 4(-\cos x) + 2 \times \frac{1}{5}x^5 - \frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1} + \underbrace{C}_{\text{Constant}}$$

$$\Rightarrow G(x) = -4\cos x + \frac{2}{5}x^5 - 2x^{1/2} + C$$

$$\boxed{G(x) = -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C}$$

Example: find f if $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$?

$$f'(x) = e^x + 20(1+x^2)^{-1} \Rightarrow f'(x) = e^x + 20\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow f(x) = e^x + 20(\tan^{-1}(x)) + C$$

Now we know that $f(0) = -2 \Rightarrow e^0 + 20(\text{Arctan}(0)) + C = -2$

$$\Rightarrow 1 + 20(0) + C = -2 \Rightarrow C = -3$$

$$\Rightarrow \boxed{f(x) = e^x + 20\tan^{-1}x - 3}$$

Example: find f, if $f''(x) = 12x^2 + 6x - 4$, and $f(0) = 4$ and $f(1) = 1$?

$$f''(x) = 12x^2 + 6x - 4 \Rightarrow f'(x) = \frac{12}{2+1}x^{2+1} + \frac{6}{1+1}x^{1+1} - 4x + C_1$$

The first antiderivative

$$\Rightarrow f'(x) = 4x^3 + 3x^2 - 4x + C_1$$

$$\Rightarrow f(x) = \frac{4}{3+1}x^4 + \frac{3}{2+1}x^{2+1} - \frac{4}{1+1}x^{1+1} + C_1 x + C_2$$

$$\Rightarrow f(x) = x^4 + x^3 - 2x^2 + C_1 x + C_2$$

$f(0) = 4$

$f(1) = 1$

$$\Rightarrow f(0) = 0^4 + 0^3 - 2(0)^2 + C_1(0) + C_2 = 4$$

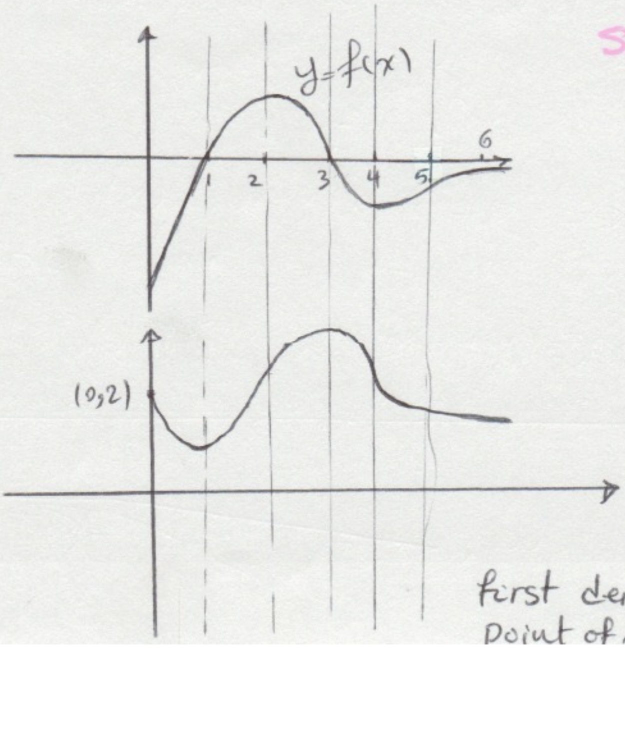
$$\Rightarrow \boxed{C_2 = 4}$$

$$f(1) = 1^4 + 1^3 - 2\times1^2 + C_1(1) + 4 = 1$$

$$f(1) = 1 + 1 - 2 + C_1 + 4 = 1 \Rightarrow \boxed{C_1 = -3}$$

$$\Rightarrow \boxed{f(x) = x^4 + x^3 - 2x^2 - 3x + 4}$$

Example: the graph of a function f is given in figure 3. Make a rough sketch of an antiderivative F, given that $F(0) = 2$.



Solution: we know that the slope of $y = F(x)$ is $f(x)$. we know that $F(0) = 2$ so we start from point $(0, 2)$, at values $x < 1$, $f(x) < 0$ or negative so we know $F(x)$ in that range is decreasing. $f(x) = 0$ at $x = -1$ and then becomes positive for $x > 1$ so $F(x)$ has a minimum wher $f(x) = 0$
at $x = 2$, $f'(x) = 0$ (or $F''(x) = 0$) so we have an inflection point there. at $x = 3$, $f(x) = 0$ so we have a Max (why? think about the first derivative test). at $x = 4$, we have another point of inflection $f'(x) = 0$, $F''(x) = 0$

Examples of Rectilinear Motion:

Example 1: A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm, find its position function $s(t)$?

solution: $v'(t) = a(t) = 6t + 4 \Rightarrow v(t) = \frac{6}{2} t^{1+1} + 4t + c_1$

$$\Rightarrow v(t) = 3t^2 + 4t + c_1$$

$$v(0) = -6 \quad \Rightarrow 0 + 0 + c_1 = -6$$

$$\Rightarrow \boxed{c_1 = -6}$$

$$\Rightarrow v(t) = 3t^2 + 6t - 6$$

$$s'(t) = v(t) = 3t^2 + 6t - 6 \Rightarrow s(t) = \frac{3}{2+1} t^{2+1} + \frac{6}{1+1} t^{1+1} - 6t + c_2$$

$$\left.\begin{array}{l} \Rightarrow s(t) = t^3 + 3t^2 - 6t + c_2 \\ s(0) = 9 \Rightarrow c_2 = 9 \end{array}\right\} \boxed{c_2 = 9}$$

$$\boxed{s(t) = t^3 + 2t^2 - 6t + 9}$$

Example: A ball is thrown upward w/ a speed of 48 ft/s from the edge of a cliff, 432 ft above the ground. Find the height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

Solution: the motion is vertical and we choose the positive direction to be upward. At time t, the velocity is decreasing.

$$a(t) = \frac{dv}{dt} = -32$$

we take the antiderivative to obtain the velocity $\Rightarrow v(t) = -32t + c_1$

we know that $v(0) = 48 \Rightarrow -32 \times 0 + c_1 = 48$

$$\Rightarrow c_1 = 48$$

$$\Rightarrow v(t) = -32t + 48$$

$$v(t) = \frac{ds}{dt} \Rightarrow s(t) = -\frac{32}{2}t^{1+1} + 48t + c_2 = -16t^2 + \quad t + c_2$$

$$s(0) = 432 \Rightarrow s(0) = 0 + 0 + c_2 = 432 \Rightarrow c_2 = 432$$

$$s(t) = -16t^2 + 48t + 432$$

when the ball hits the ground we have $s(t) = 0$

$$\Rightarrow \underbrace{-16t^2 + 48t + 432 = 0}_{\text{we divide both sides by } -16} \Rightarrow t^2 - 3t - 27 = 0 \Rightarrow \text{quadratic equation.}$$

$$t^2 - 3t - 27 = 0 \Rightarrow t = \frac{3 \pm 3\sqrt{13}}{2}$$

, we reject the solution with negative sign as it gives a negative value for t.

$$\Rightarrow t = \frac{3 + 3\sqrt{13}}{2} \approx 6.9s$$

Therefore, the ball hits the ground 6.9s later.