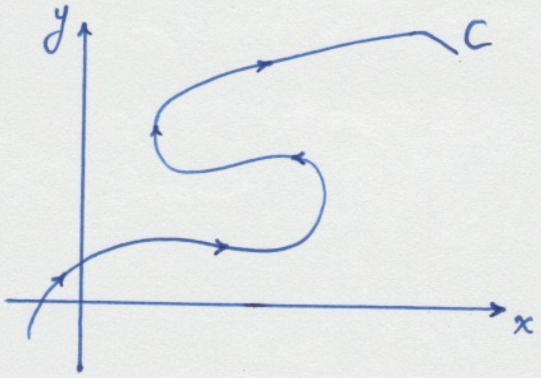
Introduction: suppose that you have a curve like C that is shown in the figure below:



we know that it is impossible to describe C by an equation of the form $y = f(x)$, because C fails the vertical line test. (it is not the graph of a function)

However, we can write x&y coordinates as a function of a third variable like t, called a **parameter**:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

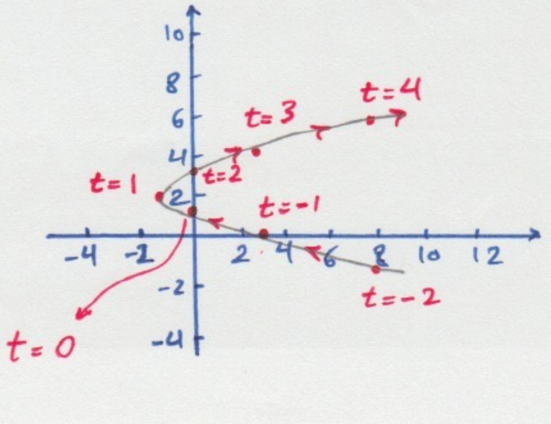
we call these equations, **Parametric equations**. we also call C, a **parametric curve**.

In other words for each value of t, we can obtain a value for the x-coordinate and a value for the y-coordinate of C.

Example: sketch and identify the curve defined by the parametric equations $x = t^2 - 2t$, and $y = t + 1$

Solution: Each of the points on the graph of the parametric curve is associated with a value of t. Therefore, we can calculate values for x and y coordinates as followed

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



Note: In the previous example, you have noticed that the graph actually has a direction (shown by the arrows). That is the direction of the curve as t values are increasing.

Note: the fact that some of the parametric curves are not a graph of a function, does not mean that we can NOT eliminate parameter t and write the eq. as a relationship b/w x and y.

e.g. in the previous example:

$$x = t^2 - 2t \longrightarrow x = t^2 - 2t = (y-1)^2 - 2(y-1)$$

$$y = t + 1 \longrightarrow t = y - 1 \qquad \Rightarrow \boxed{x = y^2 - 4y + 3}$$

This is the eq. of the curve in the previous example.

Note: In the previous example, we did not put a restriction on the values that t can take but in general, t may be bounded, say $a \le t \le b$. In these cases

$$\begin{cases} (f(a), g(a)) & \text{is called The initial point.} \\ (f(b), g(b)) & \text{is called The terminal point.} \end{cases}$$

Example: what curve is represented by the following parametric equations?

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \qquad 0 \le t \le 2\pi$$

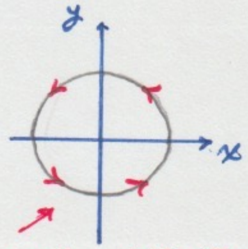
Solution: we try to find a way to eliminate t in these equation:

$$x^2 = \cos^2 t$$

$$y^2 = \sin^2 t \qquad \Rightarrow x^2 + y^2 = 1$$

$$\sin^2 t + \cos^2 t = 1$$

This is the eq. of a circle



Notice the arrows

Example: What curve is represented by the given parametric equations?

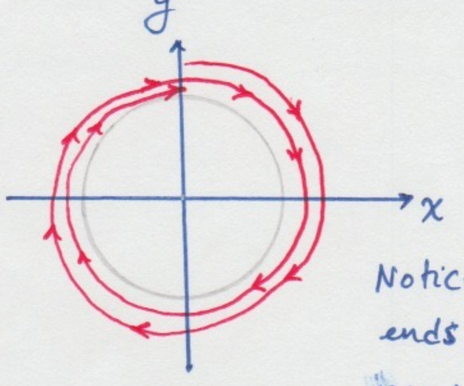
$$\begin{cases} x = \sin 2t \\ y = \cos 2t \end{cases} \quad 0 \leq t \leq 2\pi$$

Solution: Similar to the previous example we can write:

$x^2 = \sin^2 2t$

$y^2 = \cos^2 2t$ $\Rightarrow x^2 + y^2 = 1$ This is also the eq. of a circle, but what is the difference b/w the graph of this parametric eq. w/ the previous one?

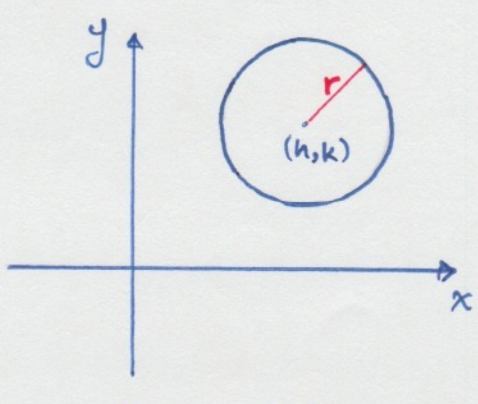
$\sin^2 2t + \cos^2 2t = 1$



Notice that in this example, t starts from 0 and ends in 2π but we move twice around the circle in the clockwise direction. This is the difference b/w this example and the previous one.

Note: Different sets of parametric equations can represent the same curve. Thus, we distinguish between a curve, which is a set of points, and a parametric curve, in which the points are traced in particular way.

Example: Find parametric equations for the circle w/ center (h,k) and radius r.



Solution: We know that the parametric eq. of a circle is $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ when the radius is 1.

If we multiply the expressions for x and y by r, we get $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$ when the radius is r

Now, if we shift the center h units in x direction and k units in y direction we get, $\begin{cases} x = h + r\cos t \\ y = k + r\sin t \end{cases}$

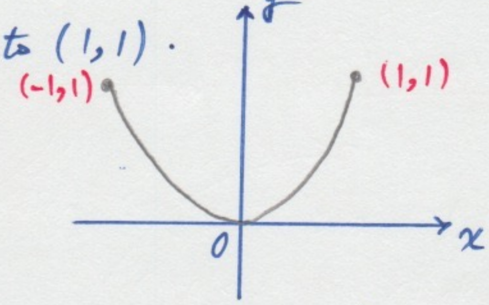
Example: Sketch the curve with parametric eq. $x = \sin t$, $y = \sin^2 t$.

Solution: $x = \sin t$
$y = \sin^2 t = \sin t \cdot \sin t = x \cdot x = x^2$

$\Rightarrow y = x^2$, therefore the grap of this parametric equations is a Parabola.

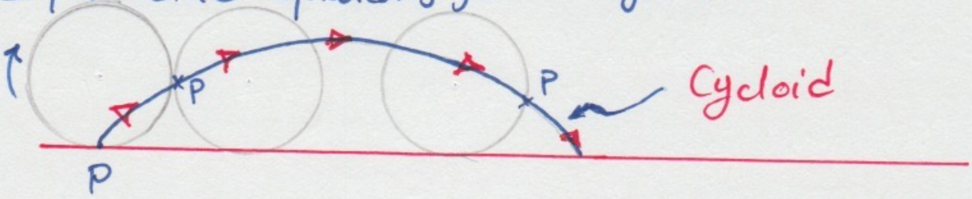
Also, Note that since $-1 \leqslant \sin t \leqslant +1 \Rightarrow -1 \leqslant x \leqslant +1$. This means, only the part of parabola, for which $-1 \leqslant x \leqslant +1$ should be represented.

Additionally, $\sin t$ is a periodic function, which means the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely along the parabola from $(-1, 1)$ to $(1, 1)$.



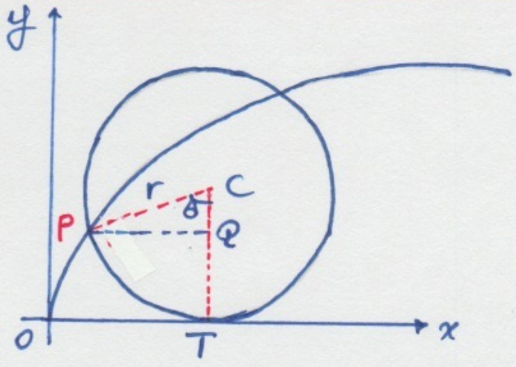
Example: The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called cycloid. If the circle has the radius r and rolls along x_axis and if one position of P is the origin, find the parametric equations for the cycloid.

Solution:



Think about it this way. Let's say you have a wheel (something that looks like a circle). You mark one point on the circumference of the wheel and you name it P. Now if you start rolling the wheel along x-axis, the path that the point P will move along, will look like a half-ellipse but its eq. is different.

This Geometric shape is called a Cycloid.



Suppose that initially, point P, coincides with O, the origin of the coordinate system. Also, suppose that when point T is tangent to the x-axis, the Angle b/w PC and CT is θ.

We know that length of arc of PT is the same as OT.
We are interested in finding coordinates of P.

$$\left.\begin{aligned} x_p &= |OT| - |PQ| \\ |OT| &= r\theta \\ |PQ| &= r\sin\theta \end{aligned}\right\} \Rightarrow x_p = r\theta - r\sin\theta \Rightarrow \boxed{x_p = r(\theta - \sin\theta)}$$

$$\left.\begin{aligned} y_p &= |TC| - |CQ| \\ |TC| &= r \\ |CQ| &= r\cos\theta \end{aligned}\right\} \ y_p = r - r\cos\theta \Rightarrow \boxed{y_p = r(1 - \cos\theta)}$$

One arc of the cycloid comes from one rotation of the circle and so is described by $0 \leqslant \theta \leqslant 2\pi$

In differential equations, you will see a lot more about this geometric shape as we investigate brachistochrone Problem and tautochrone problem.

Brachistochrone Problem: Find a curve along which a particle will slide in the shortest time under the influence of gravity.

Tautochrone problem: No matter a particle P is placed on an inverted cycloid, it takes the same amount of time to slide to the bottom.