section 10.2

In this section, we focus on applying the methods of calculus to parametric curves for problems involving tangents, areas, and arc length.

* Tangents:

If you remember from math 2A, every time that we wanted to find the slope of the tangent line on a curve, we had to take the derivative of function $y=f(x)$ and find $\frac{dy}{dx}$.

Now, suppose f and g are differentiable functions of parameter t and we want to find the tangent line on the parametric curve $x=f(t)$ and $y=g(t)$. (y should also be a differentiable function of x)

$$\Rightarrow \frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt} \quad \text{(According to the chain Rule)}, \text{ if } \frac{dx}{dt}\neq 0$$

$$\boxed{\text{Therefore: } \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

In general, it is also very useful to calculate $\frac{d^2y}{dx^2}$ (the second derivative)

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx} \quad \text{(Now we replace } y \text{ w/ } \frac{dy}{dx} \text{ in the previous formula)}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$\rightarrow$ Note That $\frac{d^2y}{dx^2}\neq\frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$

Important.

Example: A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

a) show that C has two tangents at the point $(3,0)$ and find their eqs.

b) Find the point on C where the tangent is horizontal or vertical.

c) Determine where the curve is concave upward and downward.

d) sketch the graph.

a) Notice that when we talk about point $(3,\emptyset)$, the y-value of this point is $\emptyset$ meaning if we solve $y=\emptyset$, or $t^3 - 3t = \emptyset \Rightarrow \begin{cases} t=\emptyset \\ t=\sqrt{3} \\ t=-\sqrt{3} \end{cases}$

In other words, for $t=\emptyset, \sqrt{3}$, and $-\sqrt{3}$, y is equal to $\emptyset$

we reject $t=\emptyset$ because it does not satisfy $x=t^2$

since for $t=+\sqrt{3}$ and $t=-\sqrt{3}$, the curve crosses at $(3,0)$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{2t} \Rightarrow \begin{cases} t=\sqrt{3} \Rightarrow m = \dfrac{3(\sqrt{3})^2-3}{2\sqrt{3}} = \dfrac{6}{2\sqrt{3}} = \sqrt{3} \\ t=-\sqrt{3} \Rightarrow m = \dfrac{3(-\sqrt{3})^2-3}{2(-\sqrt{3})} = -\sqrt{3} \end{cases}$$

$$\Rightarrow y - \emptyset = \sqrt{3}(x-3) \Rightarrow y = \sqrt{3}(x-3)$$

$$y - \emptyset = -\sqrt{3}(x-3) \Rightarrow y = -\sqrt{3}(x-3)$$

b) The tangent is horizontal when $\frac{dy}{dx} = \emptyset \Rightarrow \frac{dy}{dt} = \emptyset$ and $\frac{dx}{dt} \neq \emptyset$

$\frac{dy}{dt} = 3t^2 - 3 = 0 \Rightarrow t = \pm 1 \Rightarrow$ At $(\overset{x}{1}, -\overset{y}{2})$ and $(\overset{x}{1}, \overset{y}{2})$ the tangent line is horizontal

we have a vertical tangent line when $\frac{dx}{dt} = \emptyset$ and $\frac{dy}{dt} \neq \emptyset$

$\Rightarrow 2t = \emptyset \Rightarrow t = \emptyset \Rightarrow$ At the point $(\underbrace{0}_{x}, \underbrace{0}_{y})$ the tangent line is vertical.

Associated w/ $t=\emptyset$

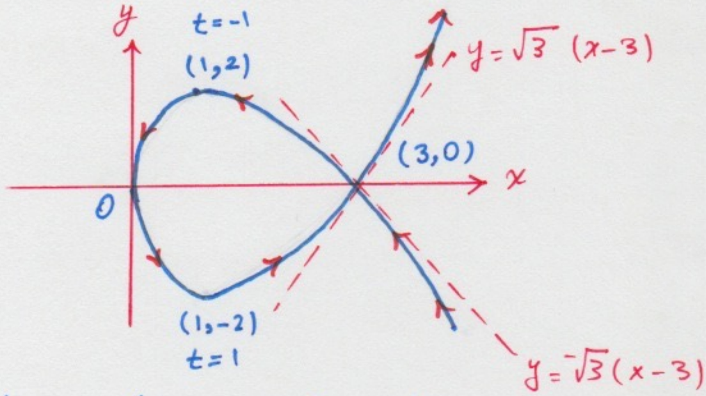
c) To determine concavity, we should Calculate $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} = \frac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t} = \frac{3(t^2+1)}{4t^3}$$

$\Rightarrow$ for all the values of $t>0$, $\frac{d^2y}{dx^2}>0$ and the curve is Concave upward.

for all the values of $t<0$ $\frac{d^2y}{dx^2}<0$ and the curve is Concave downward.

d) put all the information together:



Example: Find the tangent to the cycloid $x=r(\theta-\sin\theta)$ and $y=r(1-\cos\theta)$ at the point where $\theta=\frac{\pi}{3}$. At what point is the tangent horizontal? when is it vertical?

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1-\cos\theta)} \Rightarrow m = \frac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \sqrt{3}$$

when $\theta=\frac{\pi}{3}$ $\begin{cases} x = r\left(\frac{\pi}{3}-\sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right) \\ y = r\left(1-\cos\frac{\pi}{3}\right) = \frac{r}{2} \end{cases}$ $\Rightarrow$ Tangent line eq.

$$y-\frac{r}{2} = \sqrt{3}\left(x-\left(r\frac{\pi}{3}-\frac{r\sqrt{3}}{2}\right)\right)$$

$$\sqrt{3}x-y = \sqrt{3}\left(\frac{\pi\cdot r}{3}-\frac{r\sqrt{3}}{2}\right)-\frac{r}{2} \Rightarrow \boxed{\sqrt{3}x-y = r\left(\frac{\pi}{\sqrt{3}}-2\right)}$$

tangen is horizontal when $m=0 \Rightarrow \frac{\sin\theta}{1-\cos\theta}=0 \Rightarrow \theta=(2n-1)\pi$

remember That $1-\cos\theta\neq 0$

Therefore, the corresponding point is $((2n-1)\pi r, 2r)$

Another solution for $\sin\theta=0$ is $\Rightarrow$ $\boxed{\theta=2\pi n \rightarrow \text{There is a problem Here}}$

when $\theta = 2\pi n$, both $dx/d\theta$ and $dy/d\theta$ are Zero. (Because $\cos(2n\pi) = 1$ & $\cos\theta - 1 = 0$)

$$\Rightarrow \lim_{\theta \to 2\pi n^+} \frac{dy}{dx} = \lim_{\theta \to 2\pi n^+} \frac{\sin\theta}{1-\cos\theta} = \frac{0}{0} \longrightarrow \text{we use L'Hospital's Rule}$$

$$\longrightarrow \lim_{\theta \to 2\pi n^+} \frac{\cos\theta}{\sin\theta} = +\infty$$

$$\longrightarrow \lim_{\theta \to 2\pi n^-} \frac{\cos\theta}{\sin\theta} = -\infty$$

This means we have vertical tangent(s) when $\theta = 2n\pi$ and those are $\boxed{x = 2n\pi r}$

more than one (infinite)

Areas:

we know that the area under a curve $y = h(x)$ from a to b is $A = \int_a^b h(x)\,dx$ where $h(x) \geq 0$. in parametric curves the area formula will be converted to

$x = f(t)$

$y = g(t)$

$$\Rightarrow \int_a^b \underbrace{h(x)}_{y = h(x)} dx = \int_\alpha^\beta g(t) f'(t)\,dt$$

$y = g(t)$

$\frac{dx}{dt} = f'(t)$

$\Rightarrow dx = f'(t)\,dt$

Example: find the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

one arch of cycloid is given by $0 \leq \theta \leq 2\pi r$

$$\Rightarrow \text{Area} = \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1-\cos\theta) \cdot r(1-\cos\theta)\,d\theta = r^2 \int_0^{2\pi} (\cos^2\theta + 1 - 2\cos\theta)\,d\theta$$

$$= r^2 \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2} + 1 - 2\cos\theta\right) d\theta = r^2 \left(\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta - 2\sin\theta\right)\Big|_0^{2\pi}$$

$$= r^2 \left(\frac{3}{2} \cdot 2\pi\right) = 3\pi^2$$

Arc Length:

We already know how to find length L of a curve C given in the form of

$y = h(x)$, $a \leq x \leq b$: $\ell = \int_a^b \sqrt{1+(h'(x))^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Now suppose that curve C is described by parametric equations $x = f(t)$ $y = g(t)$ and $\alpha \leq t \leq \beta$ where $\frac{dx}{dt} = f'(t) > 0$ (This means that C is transversed once from left to right), as t increases from α to β ($a = f(\alpha)$, and $b = f(\beta)$).

$$\Rightarrow \ell = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2} \cdot \frac{dx}{dt} \cdot dt$$

$$\frac{dx}{dt} > 0 \Rightarrow \boxed{\ell = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}$$

We do NOT prove the theorem but it is important to remember that even if C can not be expressed in terms of (in the form of) $y = h(x)$, the above formula is still applicable.

Example: Find the arc length of the circle with $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$.

$$\ell = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi} dt = 2\pi$$

Example: Find the arc length of the circle with $x = \cos 2t$ and $y = \sin 2t$, $0 \leq t \leq 2\pi$.

$$\int_0^{2\pi} \sqrt{4\cos^2 t + 4\sin^2 t}\,dt = \int_0^{2\pi} 2\,dt = 4\pi$$

Both curves in this example and the previous example are circles with the same radius. Why is it that the length of the curve for one is 2π and the other one is 4π?

Example: Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$ and $y = r(1-\cos\theta)$.

$0 \leq \theta \leq 2\pi$. $\frac{dx}{d\theta} = r(1-\cos\theta)$ and $\frac{dy}{d\theta} = r\sin\theta \Rightarrow \ell = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta = \int_0^{2\pi} r\sqrt{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta \qquad 1-\cos\theta = 2\sin^2\theta/2$$

$$= r\int_0^{2\pi} \sqrt{4\sin^2\theta/2}\, d\theta = 2r\int_0^{2\pi} \sin\theta/2\, d\theta = 2r\left[-2\cos\theta/2\right]_0^{2\pi} = -4r\left[\cos\theta/2\right]_0^{2\pi}$$

$$= -4r\left[\cos\frac{2\pi}{2} - \cos 0\right] = -4r(-1-1) = \boxed{8r}$$