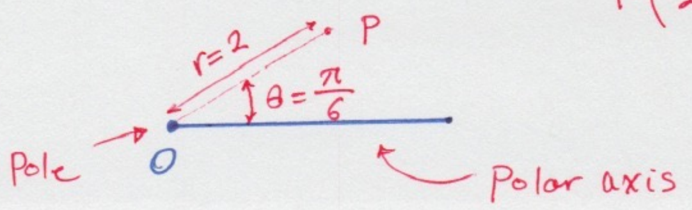
Let's think about why we use a Coordinate system!
Why do we use them?
We use coordinate systems to represent a point in the plane (or space), by an ordered pair of numbers called Coordinates. So far we have used Cartesian (rectangular) coordinate system. Cartesian coordinates are the direct distance from two perpendicular axes.

Today we want to introduce a new coordinate system that is used to represent points in a plane as well. It is called Polar Coordinate system.

Let's pick a point in the plane and label it O. We call this point the Pole. Then draw a half-line horizonally to the right of point O. Let's call this half-line the Polar axis. Now we can represent any point in the plane (say point P) using an ordered pair, (r, θ) such that r is the distance (direct distance) between point P and the Pole (point O), and θ is the angle between the polar axis and line OP. It is the convention that the angle (θ) is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction.

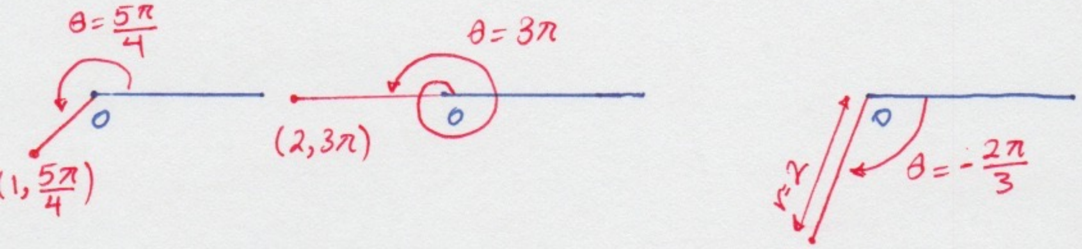
Note: we can extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that the point $(-r, \theta)$ and (r, θ) lie on the same line through O with the same distance from O but on the opposite side of O

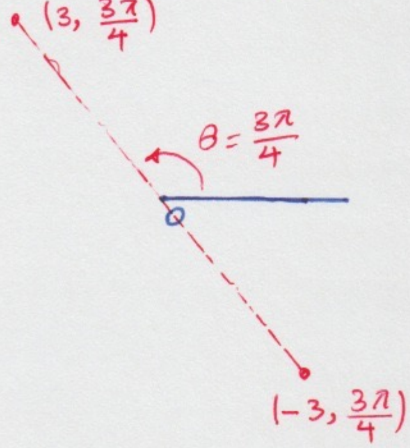
$P(2, \frac{\pi}{6})$



Example: plot the points whose polar coordinates are given.

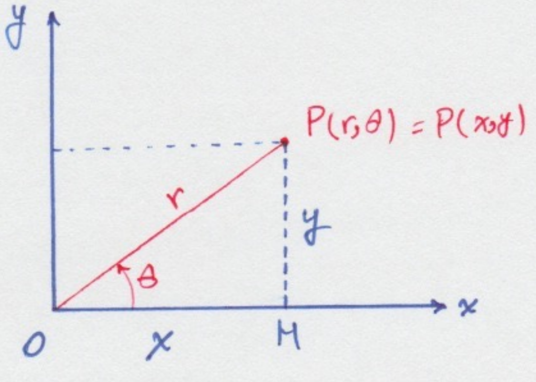
a) $(1, \frac{5\pi}{4})$ b) $(2, 3\pi)$ c) $(2, \frac{-2\pi}{3})$ d) $(-3, \frac{3\pi}{4})$





Connection between polar and Cartesian coordinates:

Now that we know polar coordinates are another way to represent points in a plane, we want to find the connection between polar coordinates and Cartesian coordinates. Let's say that we have a point in a Cartesian coordinate system, $P(x, y)$, and we want to know the coordinates of the same point in the polar coordinate system, $P(r, \theta)$. Using simple geometry, it is obvious that:



in OPH triangle we have

$$\cos\theta = \frac{x}{r} \quad , \quad \sin\theta = \frac{y}{r}$$

$$\Rightarrow \boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

using these simple formulas we can convert polar coordinates to Cartesian.

Also, $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$$\Rightarrow x^2 + y^2 = r^2$$

$$y/x = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta = \frac{y}{x}$$

$$\Rightarrow \boxed{\begin{aligned} x^2 + y^2 &= r^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}}$$

using these two simple formulas we can convert Cartesian coordinates to polar coordinates.

Example: Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian Coordinates.

$r = 2$, $\theta = \frac{\pi}{3}$

$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2 \times 1/2 = 1$

$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

} Therefore The given point is $(1, \sqrt{3})$ in the Cartesian coordinate system.

Example: Represent the point with Cartesian Coordinates $(1, -1)$ in terms of polar Coordinates.

$x = 1$, $y = -1$ $\Rightarrow x^2 + y^2 = r^2 \Rightarrow r^2 = 1^2 + (-1)^2 = 2 \Rightarrow \boxed{r = \sqrt{2}}$

$\tan\theta = \frac{y}{x} = -1 \Rightarrow \theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$ (which is $-\frac{\pi}{4}$ too)

Since $(1, -1)$ is in the fourth quadrant, we reject $\theta = \frac{3\pi}{4}$ (This is in the second quadrant)

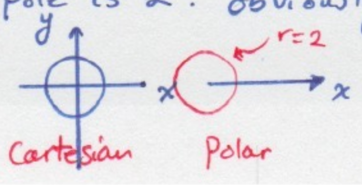
Polar Curves:

Similar to the graph of functions in the Cartesian Coordinate system, that $y = f(x)$ or more generally $F(x, y) = \emptyset$, the graph of relationships/ functions may be sketched in polar Coordinates.

The graph of a polar equation $r = f(\theta)$, or more generally $F(r, \theta) = \emptyset$ consists of all the points that have at least one polar representation (r, θ) whose Coordinates satisfy the equation.

Example: what curve is represented by the polar equation $r = 2$?

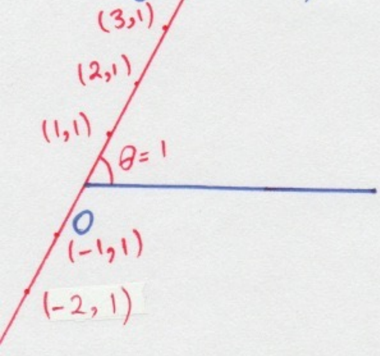
$\underline{r}$ represents the distance from the point to the pole. Therefore, $r = 2$ represents all the points in the plane That Their distance to Pole is 2. obviously, that is a circle w/ radius of 2.



* Note: The equation $r=a$ represents a circle with center O, and radius $|a|$.

Example: Sketch the polar curve $\theta = 1$.

Solution: This curve consists of all the points such that the polar angle θ is 1 radian. This would be a line.



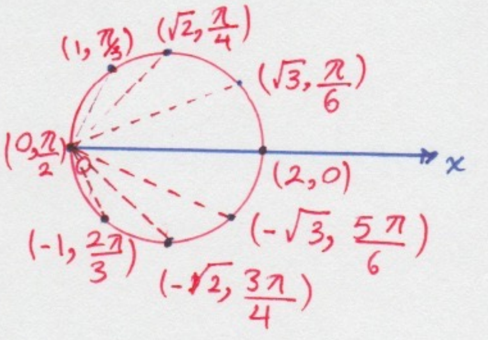
Example: Sketch the curve with polar equation $r = 2\cos\theta$

Find the Cartesian equation for this curve.

Solution: in order to plot the above relationship, we plug in some convenient values for θ and calculate the value of r. We find the points in the plane and connect them:

Part a)

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2



Part b)

We know in polar coordinates

$x = r\cos\theta$

We have $r = 2\cos\theta$

$\Rightarrow \cos\theta = r/2 = \dfrac{x}{r}$

$\Rightarrow r^2 = 2x$

$\Rightarrow 2x = r^2 = x^2 + y^2$

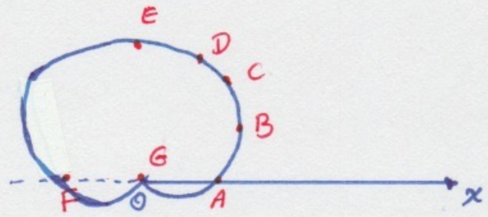
$\Rightarrow x^2 - 2x + y^2 = 0$

$\Rightarrow x^2 - 2x + 1 + y^2 = 1 \Rightarrow (x-1)^2 + y^2 = 1$

It is a circle with $r=1$ and centered at $(1,0)$

Example: Sketch the curve $r = 1 + \sin\theta$.

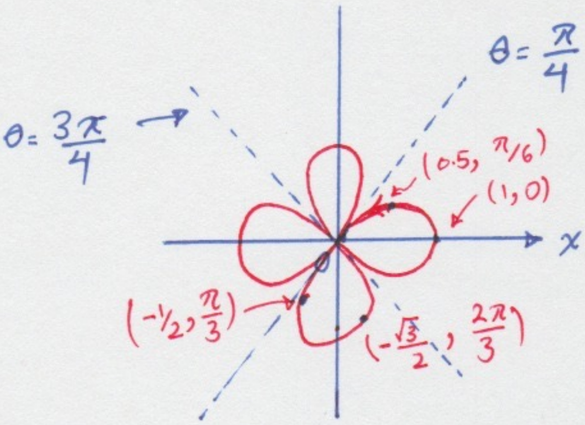
	θ	$1+\sin\theta$
A:	0	$1+\sin 0 = 1$
B:	$\pi/6$	$1+\sin\frac{\pi}{6} = \frac{3}{2} = 1.5$
C:	$\pi/4$	$1+\sin\frac{\pi}{4} = 1+\frac{\sqrt{2}}{2} \approx 1.7$
D:	$\pi/3$	$1+\sin\frac{\pi}{3} = 1+\frac{\sqrt{3}}{2} \approx 1.85$
E:	$\pi/2$	$1+\sin\frac{\pi}{2} = 1+1 = 2$
F:	π	$1+\sin\pi = 1+0 = 1$
G:	$3\pi/2$	$1+\sin\frac{3\pi}{2} = 1-1 = 0$



This polar curve in mathematics is called Cardioid (Looks like a heart)

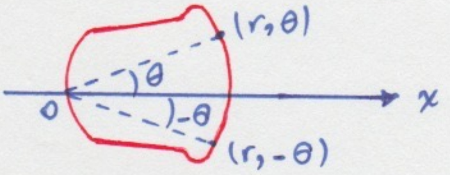
Example: Sketch the curve $r = \cos 2\theta$

θ	$\cos 2\theta$
0	1
$\pi/6$	$1/2$
$\pi/4$	0
$\pi/3$	$-1/2$
$\pi/2$	-1
$2\pi/3$	$-\sqrt{3}/2$
$\frac{3\pi}{4}$	0
π	1
$3\pi/2$	-1
2π	1

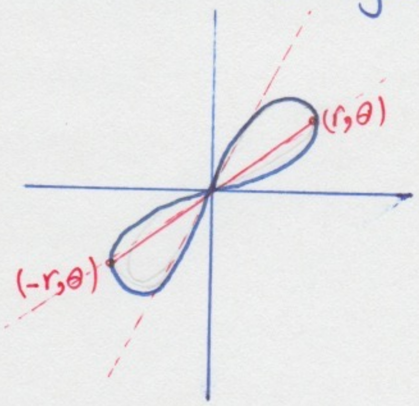


Symmetry Rules for sketching polar Coordinates:

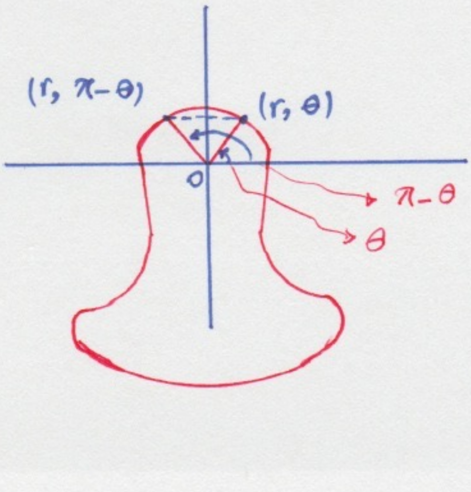
A) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.



B) If the equation is unchanged when r is replaced by $-r$ or when θ is replaced by $\theta+\pi$, the curve is symmetric about the pole. (This means the curve remains unchanged if we rotate it 180° about the origin.



c) If the equation is unchanged when θ is replaced by $\pi-\theta$, the curve is symmetric about the vertical line $\theta=\frac{\pi}{2}$ (perpendicular to the polar axis)



For Example: The curve that we sketched in one of the previous examples ($r = 2\cos\theta$) is symmetric wrt polar axis because $\cos(-\theta) = \cos\theta$.

Curves $r = 1 + \sin\theta$ and $r = \cos 2\theta$ are symmetric wrt to (about) $\theta = \frac{\pi}{2}$ because $\sin(\pi - \theta) = \sin\theta$ and $\cos 2(\pi - \theta) = \cos 2\theta$.

The four-leaved rose is also symmetric about the Pole ($r = \cos 2\theta$)

These symmetry properties are useful in sketching the curves.

Tangents to Polar Curves:

to find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equation as:

$$\left.\begin{array}{l} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left(f(\theta)\sin\theta\right)}{\frac{d}{d\theta}\left(f(\theta)\cos\theta\right)}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

I suggest you to understand This formula instead of memorizing it!

Note 1/ to find the horizontal tangents we set $\frac{dy}{d\theta} = 0$ (provided that $\frac{dx}{d\theta} \neq 0$). remember that for horizontal Tangents $m = 0 = \frac{dy}{dx}$ and $\frac{dy}{dx}$ is zero when $\frac{dy}{d\theta} = 0$ (The numerator in the above formula)

Note 2/ Likewise, to find the vertical tangents we set $\frac{dx}{d\theta} = 0$ (provided that $\frac{dy}{d\theta} \neq 0$). why? (Think about it ☺)

Note 3/ If we are trying to find tangent line at the pole, this means $r = 0$ and the above formula simflifies to $\frac{dy}{dx} = \tan\theta$ (provided $\frac{dr}{d\theta} \neq 0$)

Example: For Cardioid $r = 1 + \sin\theta$ find the slope of the tangent line when $\theta = \frac{\pi}{3}$. Find the points on the cardioid where the tangent line is horizontal or vertical.

Solution:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Therefore

$$\begin{cases} x = (1+\sin\theta)\cos\theta \\ y = (1+\sin\theta)\sin\theta \end{cases}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta\sin\theta + \cos\theta(1+\sin\theta)}{\cos^2\theta - \sin\theta(1+\sin\theta)} = \frac{\cos\theta(1+2\sin\theta)}{1-\sin^2\theta - \sin\theta - \sin^2\theta}$$

$$= \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

The slope of tangent line when $\theta = \frac{\pi}{3}$ is $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}$

$$m = \frac{\cos\frac{\pi}{3}\left(1+2\sin\frac{\pi}{3}\right)}{\left(1+\sin\frac{\pi}{3}\right)\left(1-2\sin\frac{\pi}{3}\right)} = \frac{\frac{1}{2}\left(1+2\cdot\frac{\sqrt{3}}{2}\right)}{\left(1+\frac{\sqrt{3}}{2}\right)\left(1-2\frac{\sqrt{3}}{2}\right)} = \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} = -1$$

simplify

For horizontal tangents $\frac{dy}{d\theta} = 0 \Rightarrow \cos\theta(1+2\sin\theta) = 0$

$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$1 + 2\sin\theta = 0 \Rightarrow \sin\theta = -\frac{1}{2}, \ \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

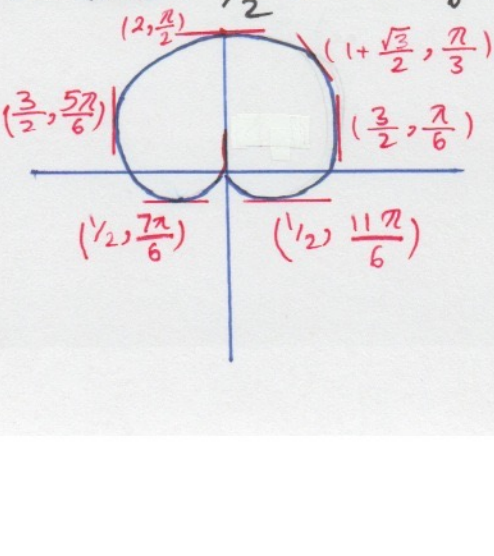
Horizontal Tangents $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

For vertical tangents $\frac{dx}{d\theta} = 0 \Rightarrow (1+\sin\theta)(1-2\sin\theta) = 0$

$\Rightarrow \sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}$

$1 - 2\sin\theta = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Vertical tangents $\theta = \frac{\pi}{6}, \frac{5\pi}{3}, \frac{3\pi}{2}$



Now, there is a problem w/ the solution of the previous example that we ignored. when $\theta = \frac{3\pi}{2}$, $\frac{dy}{d\theta} = 0$ but $\frac{dx}{d\theta}$ is also equal to zero.

$$\Rightarrow \lim_{\theta \to \frac{3\pi}{2}^-} \frac{dy}{dx} = \lim_{\theta \to \frac{3\pi}{2}^-} \left(\frac{1+2\sin\theta}{1-2\sin\theta} \right) \lim_{\theta \to \frac{3\pi}{2}^-} \frac{\cos\theta}{1+\sin\theta}$$

$$= -\frac{1}{3} \underbrace{\lim_{\theta \to \frac{3\pi}{2}^-} \frac{\cos\theta}{1+\sin\theta}}_{\to \frac{0}{0}} \xrightarrow{\text{L'Hospital}} = -\frac{1}{3} \lim_{\theta \to \frac{3\pi}{2}^-} \frac{-\sin\theta}{\cos\theta}$$

$$= -\frac{1}{3} \cdot (-\infty) = +\infty$$

Similarly $\lim_{\theta \to \frac{3\pi}{2}^+} \frac{dy}{dx} = -\infty$

This means that there is a vertical tangent line at the pole.