Three-Dimensional Coordinate Systems

Section 12.1

To represent a point in a plane, we need two numbers, (a, b), where a is the x-coordinate and b is the y-coordinate.

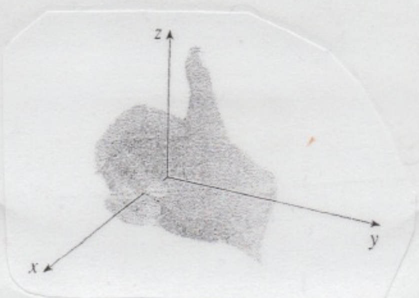
representing a point in three-dimensional space is also similar except that we need three numbers (a, b, c) or (x, y, z). a, b, and c are the x-coordinate, y-coordinate, and the z-coordinate, respectively.

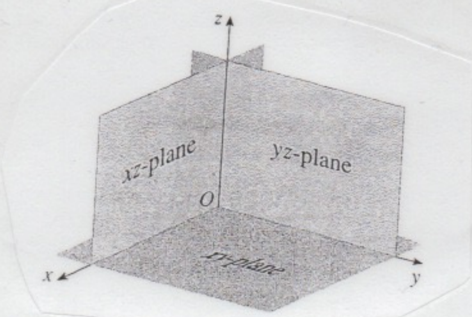
To better understand 3D-coordinate system, think about x, and y axes as horizontal axes and z as the vertical axis.

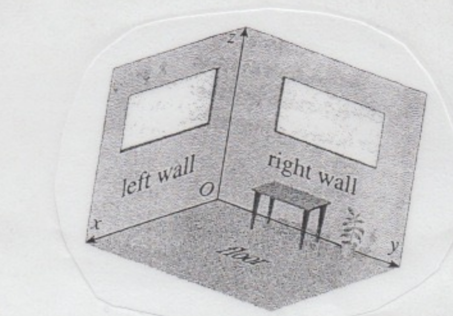
The Right-Hand Rule:

The positive direction of z-axis is determined by the right-hand Rule that states: if you curl the fingers of your right hand around the z-axis in counterclockwise direction rotation from the positive x-axis to the positive y-axis, then your thumb points in the direction of positive z-axis.

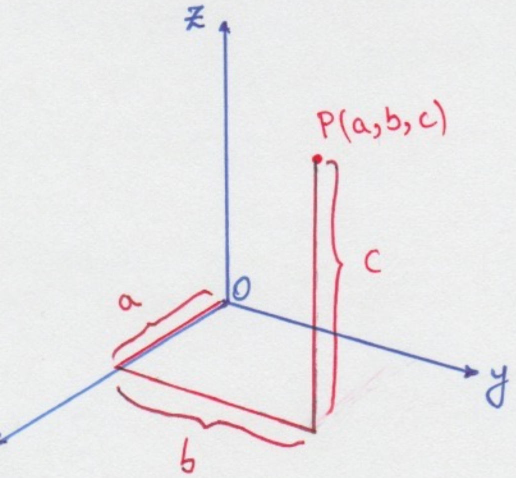
A three dimensional coordinate system creates three coordinate planes. The xy-plane is the plane that contains the x and y axes; the yz plane is the is the plane that contains y and z axes and xz plane is the plane that contains x and z axes. These three coordinate planes divide the space into eight parts, called Octants.







Let's assume that P is any Point in space w/ (a, b, c) Coordinates.



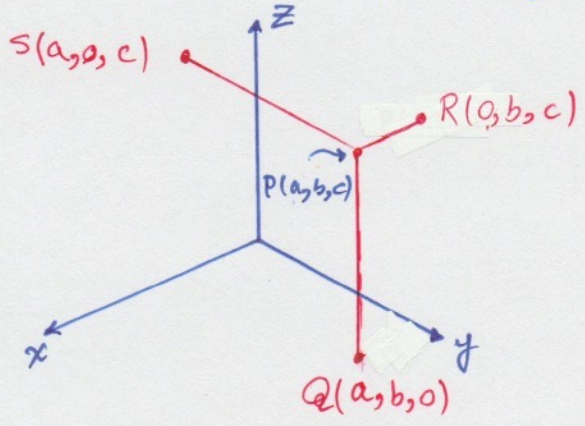
a is the direct distance from P to yz-plane

b is the direct distance from P to xz-plane

c is the direct distance from P to xy-plane

In other words, if we start from the origin and move a units towards positive x-axis, Then move b units towards positive y-axis, and The move c units towards the positive z axis, we reach to point P with (a, b, c) Coordinates.

If we draw a perpendicular line from point P to the xy plane, we get a point Q with Coordinates (a, b, 0) called projection of P onto the xy plane. Similarly, R (0, b, c) and S(a, 0, c) are the projections of P onto yz plane and xz plane, respectively.



Surfaces:

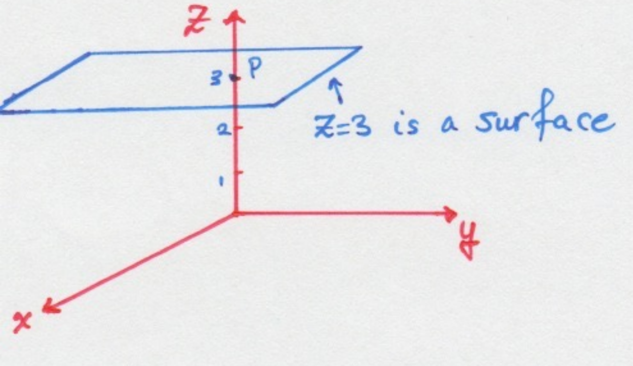
The cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. This definition is useful in sketching the surfaces in a three-dimensional rectangular coordinate system.

In a Two dimensional plane, the graph of an eq. is the relationship b/w x and y. In Three-dimensional rectangular Coordinate system, the relationship/equation b/w x, y, z represents a Surface in $\mathbb{R}^3$.

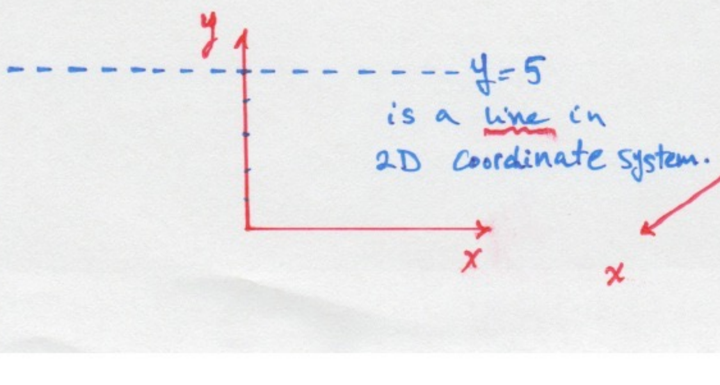
Example: Sketch the graph of the following surfaces.

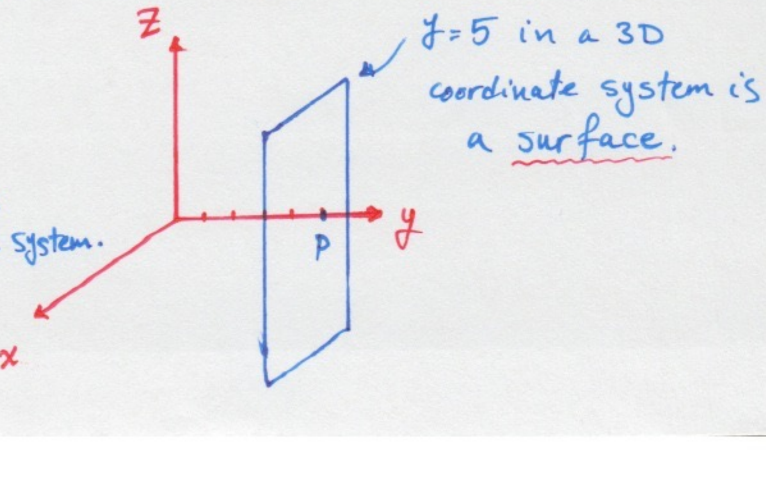
a) $z = 3$ b) $y = 5$

a) $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$ which is the set of all points in $\mathbb{R}^3$ that their z-coordinates are 3/



b) Similarly, $y = 5$ is the set of all $\{(x, y, z) \mid y = 5\}$ or all the points in $\mathbb{R}^3$ that their y-coordinates are 5.



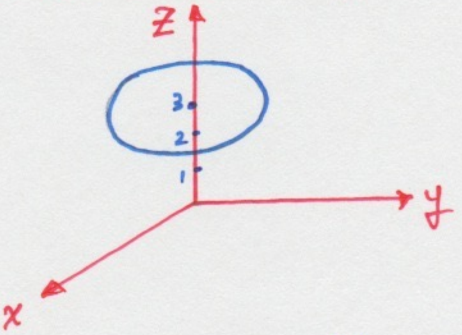


In general, $x=k$ represents a surface (plane) parallel to yz plane that intersects x-axis at k. $y=k$ represents a surface (plane) parallel to xz plane that intersects y-axis at k, And $z=k$ represent a surface parallel to xy plane that intersects z-axis at k.

Example: which surface satisfy the equation

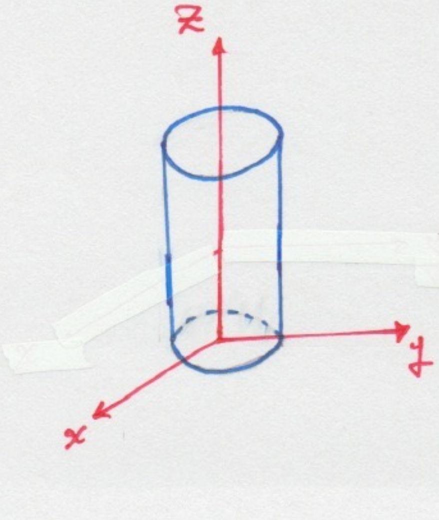
$$x^2+y^2=1 \text{ and } z=3$$

$x^2+y^2=1$, $z=3$ represents all the points such that $x^2+y^2=1$ (circle) and the distance of all these points from xy-plane is 3 ($z=3$)



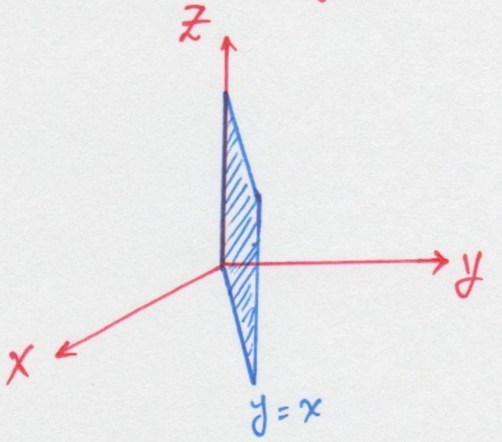
Example: which surface does the eq. $x^2+y^2=1$ represent in $\mathbb{R}^3$.

$x^2+y^2=1$ represents all the points in three-dimensional space that their $x^2+y^2=1$. This surface is composed of infinite number of circles with no restriction on z. This would be a cylinder w/ radius 1/ whose axis is the z-axis.



Example: Sketch the surface represented by $y=x$ in $\mathbb{R}^3$.

This equation represent the set of all points in $\mathbb{R}^3$ whose x and y coordinates are equal. This would be a plane that intersects xy plane in the line $y=x$ and $z=0$



Distance l between the points (x_1, y_1, z_1) and (x_2, y_2, z_2):

The distance l between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$l=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ (you can find the proof in Page 795)

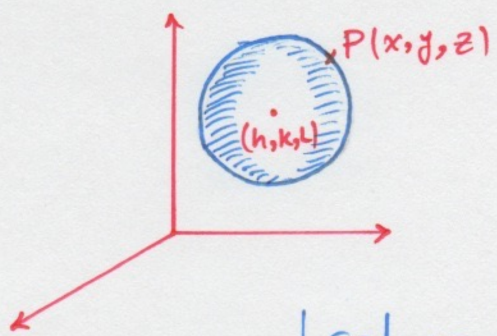
$$\Rightarrow \quad l=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Example: find the distance from the point $(2,-1,7)$ to the point $Q(1,-3,5)$.

$$|PQ|=\sqrt{(1-2)^2+(-3-(-1))^2+(5-7)^2}=\sqrt{1+4+4}=3$$

$$\Rightarrow |PQ|=3$$

Example: Find the eq. of a sphere with radius r and center $C(h,k,l)$.



Solution: By definition, a sphere is the set of all points, $P(x,y,z)$ such that their distances from the center C is r:

$$|CP| = r$$

$$|CP| = r \Rightarrow \sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2} = r$$

$$\Rightarrow \boxed{(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2} \rightarrow \text{General Formula}$$

In Particular: If the center is the origin of the coordinate system meaning, $C(0,0,0) \Rightarrow$ The equation of the sphere becomes

$$x^2 + y^2 + z^2 = r^2$$

Example: Show that $x^2+y^2+z^2+4x-6y+2z+6=0$ is the equation of a sphere, and find its center and radius.

Solution: let's complete the squares for the above eq:

$$(x^2+4x+4)+(y^2-6y+9)+(z^2+2z+1)+6-4-9-1=0$$

$$(x+2)^2+(y-3)^2+(z+1)^2 = 8$$

$$\Rightarrow r^2 = 8 \Rightarrow r = 2\sqrt{2}, \text{ The center is } (-2,3,-1).$$

Example: What region in $\mathbb{R}^3$ is represented by the following inequalities.

$$1 \leq x^2+y^2+z^2 \leq 4 \quad \text{and} \quad z \leq 0$$

Solution: The above inequality can be written as $1 \leq \sqrt{x^2+y^2+z^2} \leq 2$ this means that the above inequality represents the set of all points that their distance from the origin lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$. $z \leq 0$ means that we are restricting these points to be below the xy-plane (remember that the eq. of xy-plane in 3D coordinate system is $z=\emptyset$)