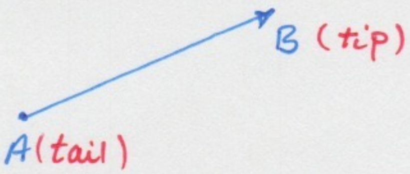
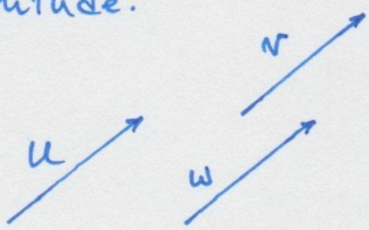
Vectors: The term vector is used to indicate a quantity that has both **magnitude & direction.** (e.g. displacement, Velocity, or force). A vector is often represented by an arrow. The length of the arrow, represents the magnitude of the vector and the arrow points in the direction of the vector.



(From point A to point B)

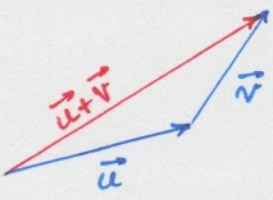
$\overrightarrow{AB}$ or $\vec{u}$

Important Note: Two vectors are equivalent (or equal) when they have the same direction, and magnitude.

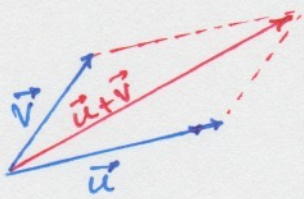


$\vec{u} = \vec{v} = \vec{w}$

Definition of Vector Addition: If $\vec{u}$ and $\vec{v}$ are vectors positioned such that the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$, then the sum $\vec{u}+\vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.

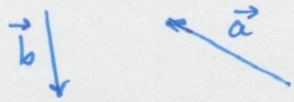


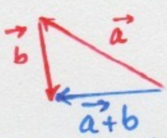
Triangle Law



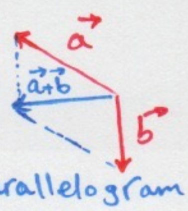
Parallelogram Law

Example: Draw the sum of vectors $\vec{a}$ and $\vec{b}$ shown below.





triangle Law

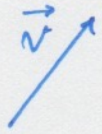


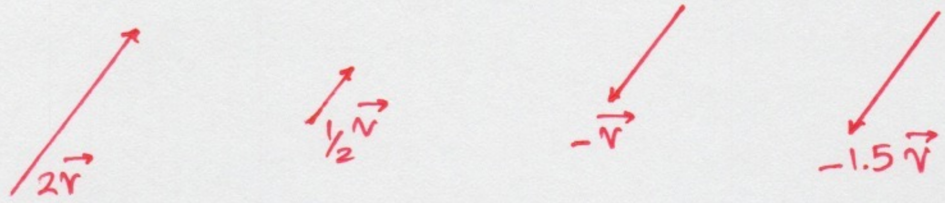
Parallelogram Law

Scalar Multiplication:

If c is a scalar (A number) and V is a vector, then the scalar multiple $c\vec{V}$ is the vector whose length is $|c|$ times the length of V and whose direction is the same as $\vec{V}$ if $c>0$ and is the opposite of $\vec{V}$ if $c<0$. If $c=0$ or $\vec{V}=0$, then $c\vec{V}=0$.

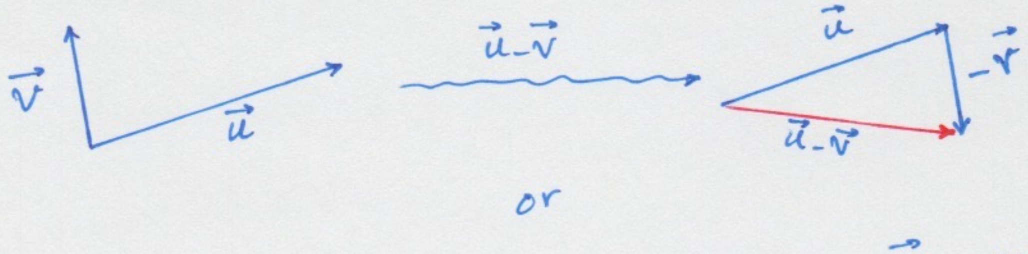
Example: suppose that we have the following vector as $\vec{V}$, find $2\vec{V}$, $\frac{1}{2}\vec{V}$, $-\vec{V}$, and $-1.5\vec{V}$.



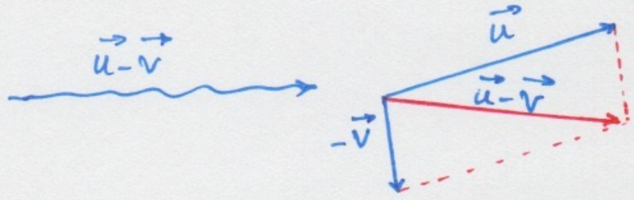


Note: Sometimes we need to subtract vectors from each other. for example, we want to draw $\vec{u}-\vec{v}$. We know that $u-v=u+(-v)$, so all we do is that we add u to scalar multiple of $\vec{V}$ and (-1).

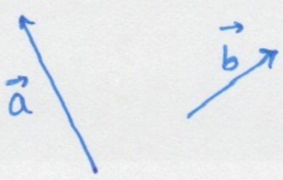
e.g.



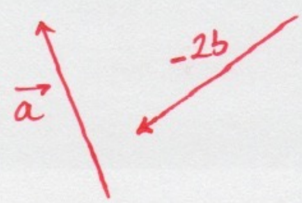
or



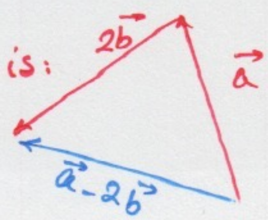
Example: If $\vec{a}$ and $\vec{b}$ are vectors shown below, draw $\vec{a}-2\vec{b}$.



$\vec{a}-2\vec{b}$ is



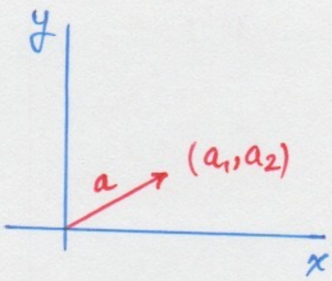
which is:

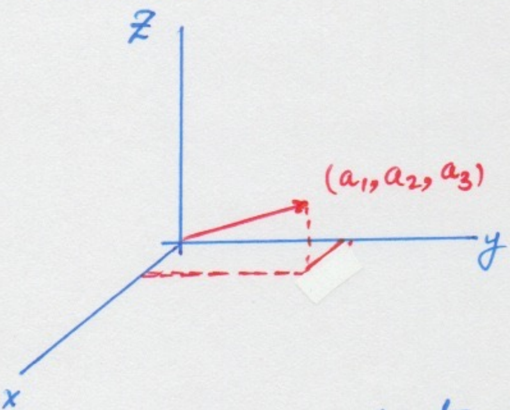


Vector components:

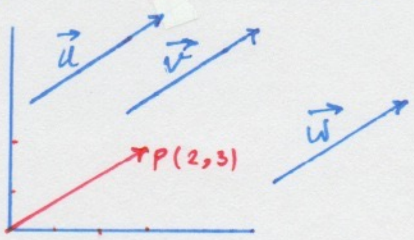
Almost always, in mathematics we treat vectors algebraically, by introducing a Coordinate system. If we place the initial point of a vector $\vec{a}$ at the origin of a rectangular coordinate system, then the terminal point of $\vec{a}$ has coordinates of (a_1, a_2) or (a_1, a_2, a_3) depending on whether our coordinate system is two or three dimensional. These coordinates are (**components**) of vector $\vec{a}$.

$\vec{a} = \langle a_1, a_2 \rangle$ or $\vec{a} = \langle a_1, a_2, a_3 \rangle$





for example, all the following vectors are equivalent to $\vec{OP}\langle 3, 2 \rangle$



what all These vectors have in common, is that if, we move The vectors such That Their initial point is on The origin, Their terminal point will reach to The point (3, 2)

* All these geometric representations can be represented algebraically as $a = \langle 3, 2 \rangle$. The particular representation $\vec{OP}$ (That starts from The origin is called The position vector of the point P.

Note: In general, in three dimensional rectangular coordinate system (this is true for 2D plane as well) Given the point $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\vec{a}$ with representation $\vec{AB}$ is

$$\boxed{a = \vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle} \rightarrow \text{Very important.}$$

Example: Find the vector represented by the direct line segment w/ initial point $A(2, -3, 4)$ and terminal point $(-2, 1, 1)$

$$a = \langle -2-2, 1-(-3), 1-4 \rangle = \langle -4, 4, -3 \rangle$$

Magnitude or length of a vector:

It is defined by $|v|$ or $\|v\|$. By using the distance formula to compute the length of a segment OP, we obtain the following formulas:

in 2D $|a| = \sqrt{a_1^2 + a_2^2}$ for $\vec{a} = \langle a_1, a_2 \rangle$

in 3D $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ for $a = \langle a_1, a_2, a_3 \rangle$

Vector Addition (Algebraically):

* To add vectors algebraically, we add corresponding components. If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then the sum is $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$

* To subtract vectors, we subtract corresponding components.

$$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

* To multiply a scalar by a vector algebraically, we multiply the number (scalar) by each component of the vector: $c\vec{a} = \langle ca_1, ca_2 \rangle$

* **Important Note:** All the above rules are true for vectors in three-dimensional space as well.

Example: If $a = \langle 4, 0, 3 \rangle$ and $b = \langle -2, 1, 5 \rangle$, find $|a|$ and the vectors $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $3\vec{b}$ and $2\vec{a} + 5\vec{b}$.

solution: $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$

$\vec{a} + \vec{b} = \langle 4 + (-2), 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle$

$\vec{a} - \vec{b} = \langle 4 - (-2), 0 - 1, 3 - 5 \rangle = \langle 6, -1, -2 \rangle$

$3\vec{b} = \langle 3(-2), 3(1), 3(5) \rangle = \langle -6, 3, 15 \rangle$

$2\vec{a} + 3\vec{b} = \langle 2 \times 4 + 5(-2), 2(0) + 5(1), 2(3) + 5(5) \rangle = \langle -2, 5, 31 \rangle$

Properties of Vectors:

The followings are some of the properties of vectors that are helpful when dealing with vectors:

If $\vec{u}$ and $\vec{v}$ and $\vec{w}$ are vectors in V_n (n-dimensional vector space) and $\underline{c}$ and $\underline{d}$ are scalars, then

* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
* $\vec{u} + \vec{0} = \vec{u}$
* $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
* $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
* $\vec{u} + (-\vec{u}) = \vec{0}$
* $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
* $(cd)\vec{u} = c(d\vec{u})$
* $1\vec{u} = \vec{u}$

* These properties of vectors can be verified either geometrically or algebraically.

e.g. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ $\longrightarrow$ Proof

$\vec{u} + \vec{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$

$= \langle v_1 + u_1, v_2 + u_2 \rangle = \langle v_1, v_2 \rangle + \langle u_1, u_2 \rangle$

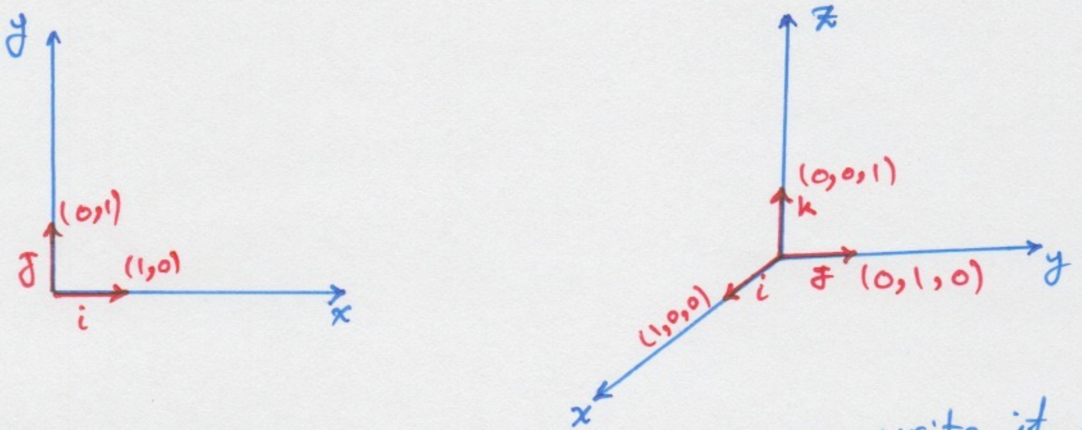
$= \vec{v} + \vec{u}$

standard basis vectors:

we define standard basis vectors, i, j, and k as vectors that have length 1 and point in the direction of positive x-, y-, and z-axis. Therefore

$\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$

Similarly, standard basis vectors i, j can be defined in a Two dimensional space: $i = \langle 1, 0 \rangle$ and $j = \langle 0, 1 \rangle$



If we have vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ we can write it as:

$$\vec{a} = a_1 \langle 1,0,0 \rangle + a_2 \langle 0,1,0 \rangle + a_3 \langle 0,0,1 \rangle$$

$\Rightarrow \boxed{\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}}$, therefore, any vector can be expressed in terms of i, j, and k.

for example: $\langle 1, -2, 6 \rangle = \vec{i} - 2\vec{j} + 6\vec{k}$

* Similarly in Two dimensions $a = \langle a_1, a_2 \rangle = a_1 \vec{i} + a_2 \vec{j}$

Example: If $\vec{a} = i + 2j - 3k$ and $\vec{b} = 4i + 7\vec{k}$, express the vector $2\vec{a} + 3\vec{b}$ in terms of $\vec{i}, \vec{j}$, and $\vec{k}$.

$$2\vec{a} + 3\vec{b} = 2(i + 2j - 3k) + 3(4i + 7k) = 2i + 4j - 6k + 12i + 21k$$

$$= 14i + 4j + 15k$$

Unit Vectors: A unit vector is a vector whose length is 1. $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors. In general if $\vec{a} \neq 0$, then $\vec{u}$ is a unit vector if

$$\vec{u} = \frac{\vec{a}}{|a|} \quad \text{why? Let } c = \frac{1}{|a|} \Rightarrow \vec{u} = c\vec{a}$$

This is always positive, Therefore, $\vec{u}$ and $\vec{a}$ are in the same direction

$$\Rightarrow |\vec{u}| = |c\vec{a}| = |c| \cdot |\vec{a}|$$

$$\Rightarrow |\vec{u}| = |c| \cdot |\vec{a}| = \frac{1}{|\vec{a}|} \cdot |\vec{a}| = 1$$

, so we just proved $|\vec{u}| = 1$, so $\vec{u}$ is a unit vector and this means that $\frac{\vec{a}}{|a|}$ is a unit vector.

Example: Find the unit vector in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

$|2i - j - 2k| = \sqrt{4+1+4} = 3$ why? because $2i - j - 2k = \langle 2, -1, -2 \rangle$

$$\Rightarrow |a| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{4+1+4}$$

$$\vec{u} = \frac{\vec{a}}{|a|} = \frac{2i - j - 2k}{3} = \frac{2}{3} i - \frac{1}{3} j - \frac{2}{3} k$$

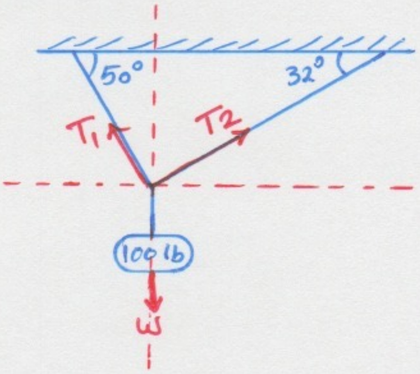
Application of Vectors:

Vectors are useful in many aspects of physics and engineering. They describe velocity, and acceleration of moving objects in space.

Another example of a vector quantity in physics is Force.

A force is represented by a vector because it has both a magnitude and a direction. If several forces are acting on an object, the resultant force experienced by the object is the vector sum of the forces.

Example: A 100-lb weight hangs from two wires as shown below. Find the tension (Force) T_1, and T_2 in both wires and also the magnitude of the tension.



* we have two forces acting on the wires T_1, and T_2. First, we express T_1 and T_2 in terms of horizontal and vertical components.

$$T_1 = -|T_1| \cos(50°)\,\vec{i} + |T_1| \sin(50°)\,\vec{j}$$

$$T_2 = |T_2| \cos(32°)\,\vec{i} + |T_2| \sin(32°)\,\vec{j}$$

we also know that $W = -100\,\vec{j}$ (because the vector is downward)

$$\vec{T} + \vec{T_2} = -W$$

$$\vec{T_1} + \vec{T_2} = 100\,\vec{j}$$

$$\left[-|T_1|\cos(50°) + |T_2|\cos(32°)\right]\vec{i} + \left[|T_1|\sin(50°) + |T_2|\sin(32°)\right]\vec{j} = 100\,\vec{j}$$

$$\Rightarrow \begin{cases} -|T_1|\cos(50°) + |T_2|\cos(32°) = 0 \\ |T_1|\sin(50°) + |T_2|\sin(32°) = 100 \end{cases} \Rightarrow \begin{cases} |T_1| = 85.64 \text{ lb} \\ |T_2| = 64.91 \text{ lb} \end{cases}$$