So far, we have seen how to add/subtract two vectors. We have also learned how to multiply a vector by a scalar. the question arises, is that how can we multiply two vectors such that their product is a useful quantity. one of these useful products is the dot product. the other is the cross-product that we introduce in the next section.

Definition of dot product:

if $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product of $\vec{a}$ and $\vec{b}$ is the number that is given by:

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

In other words, to find the dot product of $\vec{a}$ and $\vec{b}$ we multiply the corresponding components and add.

* Very Important Note: The result of dot product is Not a vector. it is a real number, that is, a scalar. sometimes we call the dot product, the scalar product or the inner product.

The above definition is given for three-dimensional vectors but we have a similar definition in Two-dimensional Vectors:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

e.g. $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$

$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = (-1)6 + 7(2) + 4(-\frac{1}{2}) = -6 + 14 - 2 = 6$

$(i + 2j - 3k) \cdot (2j - k) = 1(0) + 2(2) + (-3)(-1) = 4 + 3 = 7$

Properties of the dot product:

The dot product, obeys many laws of ordinary products of real numbers:

$\vec{a}\cdot\vec{a} = |a|^2$

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

$(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$

$\underbrace{\vec{0}}_{\text{vector}}\cdot\vec{a} = \underbrace{0}_{\text{number}}$

These properties can be easily proved:

e.g. $\vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + a_3^2 = |a|^2$

e.g. $\vec{a}\cdot(\vec{b}+\vec{c}) = \langle a_1, a_2, a_3\rangle \cdot \langle b_1+c_1, b_2+c_2, b_3+c_3\rangle$

$= a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$

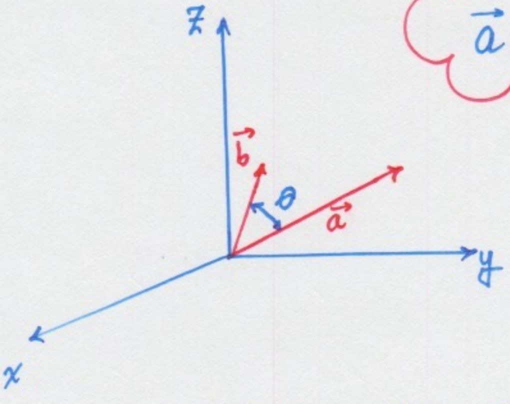
$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$

$= \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

Very important Theorem: If $\theta$ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a}\cdot\vec{b} = |a||b|\cos\theta$$

The proof of this Theorem is in Page 808 of your book.




Example: If the vectors $\vec{a}$ and $\vec{b}$ have lengths 4, and 6 and the angle b/w them is $\frac{\pi}{3}$, find $\vec{a}\cdot\vec{b}$

Solution $\vec{a}\cdot\vec{b} = |a|\cdot|b|\cos\theta = 4\times6\times\cos\frac{\pi}{3} = 24\times\frac{1}{2} = 12$

Corollary of the theorem:

If $\theta$ is the angle between the non-zero vectors $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|a|\,|b|}$$

Example: Find the angle b/w vectors $a = \langle 2, 2, -1\rangle$ and $b = \langle 5, -3, 2\rangle$.

Solution: $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{4+4+1} = 3$

$|b| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{25+9+4} = \sqrt{38}$

$\vec{a}\cdot\vec{b} = 2(5) + 2(-3) + (-1)2 = 10 - 6 - 2 = 2$

$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|a|\,|b|} = \frac{2}{3\sqrt{38}} \Rightarrow \theta = \text{Arccos}\left(\frac{2}{3\sqrt{38}}\right) = 84^\circ = 1.46 \text{ rad.}$

* Definition of orthogonal vectors:

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are called perpendicular or orthogonal if the angle b/w them is $\theta = \frac{\pi}{2}$, which means: $\vec{a}\cdot\vec{b} = 0$

* Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if $\vec{a}\cdot\vec{b} = 0$

Example: show that $2i + 2j - k$ is perpendicular to $5i - 4j + 2k$.

$(2i + 2j - k)\cdot(5i - 4j + 2k) = 10 - 8 - 2 = 0$

Therefore, the two vectors are orthogonal.

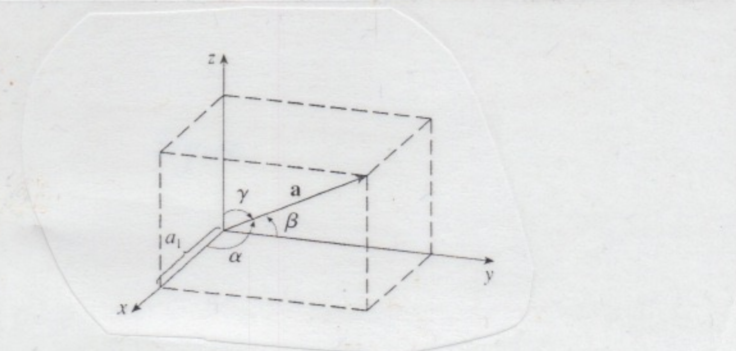
Remember if $0 < \theta < \frac{\pi}{2}$ ($\theta$ is acute) $\Longrightarrow \vec{a}\cdot\vec{b} > 0$

if $\theta = \frac{\pi}{2}$ $\Longrightarrow \vec{a}\cdot\vec{b} = 0$

if $\frac{\pi}{2} < \theta < \pi$ ($\theta$ is obtuse) $\Longrightarrow \vec{a}\cdot\vec{b} < 0$

## Direction Angles and Direction Cosines:

The direction angles of a non-zero vector $\vec{a}$ are the angles $\alpha$, $\beta$, and $\gamma$ (in the interval $[0, \pi]$) that $\vec{a}$ makes with positive $x$-, $y$-, and $z$- axes, respectively. The cosine of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the direction cosines of the vector $\vec{a}$.




$$\cos\alpha = \frac{\vec{a} \cdot i}{|a||i|} = \frac{a_1}{|a|}$$

$$\cos\beta = \frac{\vec{a} \cdot \vec{J}}{|a|\,|J|} = \frac{a_2}{|a|}$$

$$\cos\gamma = \frac{\vec{a} \cdot \vec{J}}{|a|\,|k|} = \frac{a_3}{|a|}$$

We know that $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ $\Rightarrow$ $\cos^2\alpha = \dfrac{a_1^2}{\left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2}$, $\cos^2\beta = \dfrac{a_2^2}{\left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2}$

$$\cos^2\gamma = \frac{a_3^2}{\left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2}$$

$$\Rightarrow \boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

Also we know: $\vec{a} = \langle a_1, a_2, a_3 \rangle = \langle |a|\cos\alpha, |a|\cos\beta, |a|\cos\gamma \rangle$

$$= |a| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

$$\Rightarrow \frac{\vec{a}}{|a|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

* This says, the direction cosines of $\vec{a}$ are the components of the unit vector in the direction of $\vec{a}$

Example: Find the direction angles of the vector $\vec{a} = \langle 1, 2, 3 \rangle$.

Since $|a| = \sqrt{1+4+9} = \sqrt{14}$

$$\cos\alpha = \frac{1}{\sqrt{14}}, \quad \cos\beta = \frac{2}{\sqrt{14}}, \quad \cos\gamma = \frac{3}{\sqrt{14}}$$

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74^\circ \quad \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ \quad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 37^\circ$$

## Projections:

As shown below, $\vec{PQ}$ and $\vec{PR}$ are the representation of two vectors $\vec{a}$, and $\vec{b}$ w/ the same initial point P. If S is the foot of the perpendicular from R to the line containing $\vec{PQ}$, then the vector w/ representation $\vec{PS}$ is called the <u>vector projection</u> of $\vec{b}$ onto $\vec{a}$ and is denoted by $\text{proj}_a \vec{b}$ (You can think of it as a shadow of b)

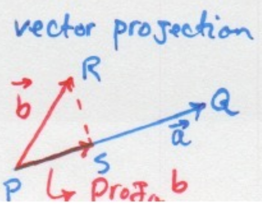
The Scalar Projection of $\vec{b}$ onto $\vec{a}$ (also called the component of $\vec{b}$ along $\vec{a}$) is defined to be the signed magnitude of the vector projection, which is $|b|\cos\theta$, where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$. This is denoted by $\text{Comp}_a \vec{b}$

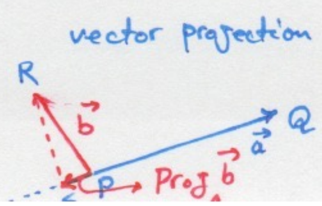
Scalar Projection of $\vec{b}$ onto $\vec{a}$ $\qquad \text{Comp}_a b = \dfrac{\vec{a}\cdot\vec{b}}{|a|}$

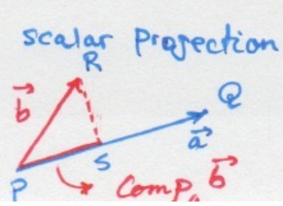
Vector projection of $\vec{b}$ onto $\vec{a}$ $\qquad \text{proj}_a b = \left(\dfrac{\vec{a}\cdot\vec{b}}{|a|}\right)\cdot\dfrac{\vec{a}}{|a|} = \dfrac{\vec{a}\cdot\vec{b}}{|a|^2}\vec{a}$

* Note that the component of $\vec{b}$ along $\vec{a}$ can be computed by taking the dot product of b w/ the unit vector in the direction of a.

* Note that the vector projection is the scalar projection times the unit vector in the direction of $\vec{a}$.



Example: Find the scalar projection and vector projection of $\vec{b} = \langle 1,1,2 \rangle$ onto $\vec{a} = \langle -2,3,1 \rangle$.

$|a| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$ $\quad$ $\text{Comp}_a \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|a|} = \frac{-2(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$

$$\text{Proj}_a \vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|b|}\right)\cdot\frac{\vec{a}}{|a|} = \frac{3}{\sqrt{14}}\left(\frac{\langle -2, +3, 1\rangle}{\sqrt{14}}\right) = \frac{3}{14}\langle -2,3,1\rangle$$

$$= \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle$$

* physical Application: The work done by the constant force, $\vec{F}$ is the dot product of $\vec{F}\cdot\vec{D}$ where D is the displacement vector.

Example: A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of wagon is held at angle 35° above the horizontal. Find the work done by the force.

$$W = \vec{F}\cdot\vec{D} = |F||D| \cos\theta = 70^{N} \cdot 100^{m} \cdot \cos 35° = 5734 \text{ N.m}$$

$$= 5734 \text{ J}$$

Example: A force given by the vector $\vec{F} = 3i + 4j + 5k$ and moves a particle from the point $P(2,1,0)$ to the point $Q(4,6,2)$. Find the work done by the force.

Displacement vector $\vec{D} = \vec{PQ} = \langle 4-2, 6-1, 2-0\rangle = \langle 2,5,2\rangle$

$W = F\cdot D = \langle 3,4,5\rangle \cdot \langle 2,5,2\rangle = 6 + 20 + 10 = 36$ (The unit depends on the unit of $\vec{F}$ and $\vec{D}$). Assuming the unit of F is N, and the unit of D is m, then the work done is 36 N.m or 36 Joules.