Given two non-zero vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, it is very useful to be able to find a non-zero vector like $\vec{c}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$. This vector ($\vec{c}$) is called the cross-product of $\vec{a}$ and $\vec{b}$ and is denoted by $\vec{a} \times \vec{b}$.

Definition: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the cross-product of $\vec{a}$ and $\vec{b}$ is a vector $\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$

Notice that the cross product is a vector unlike the dot product. For this reason, the cross product of two vectors is also called the vector product.

* Note that $\vec{a} \times \vec{b}$ is only defined when $\vec{a}$ and $\vec{b}$ are three-dimensional vectors.

The above definition of cross product of two vectors, is not easy to remember. Therefore, we try to use the notation of determinants.

A determinant of order 2 is defined by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ (multiply across and subtract)

Example: Find the determinant below.

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - (1)(-6) = 8 + 6 = 14$$

A determinant of order 3 can be defined in terms of second order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

* observe that each term on the right side of the above equation involves a number a_i in the first row of the determinant.

Note that also a_i is multiplied by the second order determinant obtained from the left side by deleting the row and column in which a_i appears. Pay attention to the minus sign in the second term of the equation.

Example: Calculate the following determinant.

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= \underbrace{1(0\times 2-4)}_{-4} - 2\underbrace{(3\times 2-(1)(-5))}_{-22} - 1\underbrace{(3\times 4-(-5)(0))}_{-12} = -38$$

If we now rewrite the definition of cross-product of two vectors using the determinant notation and the standard basis vectors i, j, k we see that the cross product of the vectors $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ is:

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

This is the formula that you should learn, instead of the previous one.

Example: if $\vec{a} = \langle 1, 3, 4 \rangle$ and $\vec{b} = \langle 2, 7, -5 \rangle$, find $\vec{a}\times\vec{b}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = i\begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - j\begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + k\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$$

$$= i(-15-28) - j(-5-8) + k(7-6) = \underline{-43i + 13j + k}$$

Example: show that $\vec{a} \times \vec{a} = 0$ for any vector $\vec{a}$ in V_3.

$$\vec{a} \times \vec{a} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = i\,(a_2 a_3 - a_3 a_2) - j\,(a_1 a_3 - a_3 a_1) + k\,(a_1 a_2 - a_2 a_1)$$

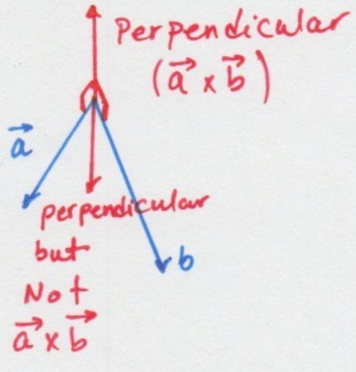
$$= i\,(0) - j\,(0) + k\,(0) = \vec{0}$$

* **Very Important Note:** We construct the cross product $\vec{a} \times \vec{b}$ so that it would be perpendicular to both $\vec{a}$ and $\vec{b}$. This is one of the most important properties of a cross product.

* $\boxed{\textbf{Theorem: } \text{The vector } \vec{a} \times \vec{b} \text{ is orthogonal to both } \vec{a} \text{ and } \vec{b}}$

The proof is simple. construct $(\vec{a} \times \vec{b}) \cdot \vec{a}$ and $(\vec{a} \times \vec{b}) \cdot \vec{b}$ and show that they are both equal to zero (remember from the previous section, two vectors are orthogonal if their dot product is zero). This means if we prove that the dot product of $\vec{a} \times \vec{b}$ and $\vec{a}$ or $\vec{b}$ is zero, they must be perpendicular/orthogonal.

Note: when we say that the cross product of $\vec{a}$ and $\vec{b}$ or $\vec{a} \times \vec{b}$ is a vector perpendicular to $\vec{a}$ and $\vec{b}$, we still do not know the direction of this vector (we can have two perpendicular vectors to both $\vec{a}$ and $\vec{b}$)



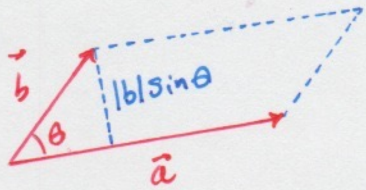
It turns out that the direction of $\vec{a} \times \vec{b}$ is given by the right-hand rule: If the fingers of your right hand curl in the direction of rotation (through an angle less than 180°) from $\vec{a}$ to $\vec{b}$ then your thumb points in the direction of $\vec{a} \times \vec{b}$

Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$ (so $0 \leq \theta \leq \pi$), then

$$|a \times b| = |a||b| \sin\theta$$

Corollary: Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = 0$

Corollary: The length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$



Area = length × height

$$= |\vec{a}| \cdot |\vec{b}| \sin\theta = |\vec{a} \times \vec{b}|$$

Example: Find a vector perpendicular to the plane that passes through the points $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Therefore, $\overrightarrow{PQ} \times \overrightarrow{PR}$ is also perpendicular to the plane that contain P, Q, and R.

$$\overrightarrow{PQ} = \langle -2-1, 5-4, -1-6 \rangle = \langle -3, 1, -7 \rangle$$

$$\overrightarrow{PR} = \langle 1-1, -1-4, 1-6 \rangle = \langle 0, -5, -5 \rangle$$

We calculate the cross-product of the vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = i\left(-5 - (-5)(-7)\right) - j\left(-3(-5) - (0)(-7)\right) + k\left((-3)(-5) - 1(0)\right)$$

$$= -40i - 15j + 15k$$

Therefore the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$, is also perpendicular to the plane.

Example: Find the area of the triangle w/ vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

These are exactly the same points as in the previous example. suppose That we have the following schematics: we know that the area of The triangle $P\hat{Q}R$ is half of the area of the parallelogram. we also know that $|\overrightarrow{PQ} \times \overrightarrow{PR}|$ = the area of The parallelogram.

$$\Rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle \Rightarrow |\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-40)^2 + (-15)^2 + (15)^2} = 5\sqrt{82}$$

therefore, the area of The triangle $= \frac{5}{2}\sqrt{82}$

If we apply the definition of cross product on the standard basis vectors i, j, and $\vec{k}$, we will have the following relationships:

$i \times j = k \quad , \quad j \times i = -k$

$j \times k = i \quad , \quad k \times j = -i$

$k \times i = j \quad , \quad i \times k = -j$

Important result: $i \times j \neq j \times i$

This means that cross product is not Commutative.

Another important result: The associative law for multiplication does not usually hold and in general $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

e.g. $\left.\begin{array}{l} i \times (i \times j) = i \times k = -j \\ (i \times i) \times j = 0 \times j = 0 \end{array}\right\}$

Properties of the cross product:

If $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors and c is scalar, then

1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2) $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Dot product.

Dot product

scalar multiplication

$\vec{a} \cdot (\vec{b} \times \vec{c})$: Scalar triple product

$\vec{a} \times (\vec{b} \times \vec{c})$: vector triple product

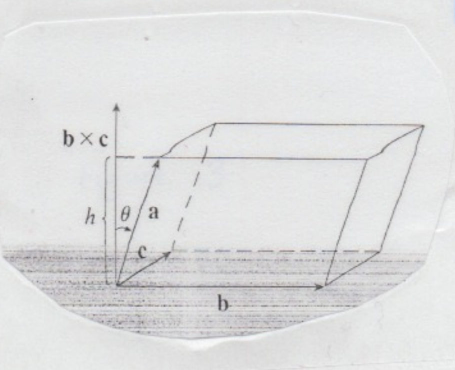
These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product / Dot product.

Triple products:

The product $\vec{a} \cdot (\vec{b} \times \vec{c})$ that occurs in property 5 is called the scalar triple product of the vectors $\vec{a}, \vec{b}$, and $\vec{c}$. we can calculate $\vec{a} \cdot (\vec{b} \times \vec{c})$ using the following determinant:

$$\vec{a} \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

. The geometric significance of the triple product is that it is used to calculate the volume of the parallelepiped created by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$



$$V = Ah = \underbrace{|b \times c|}_{\text{The area of Parallelogram made by } \vec{b} \text{ and } \vec{c}} \cdot \underbrace{|a| \cos\theta}_{\text{The height of the parallelepiped.}} = |a \cdot (\vec{b} \times \vec{c})|$$

* The volume of the parallelepiped determined by the vectors a, b, and c is the magnitude of their scalar triple product:

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

Example: use the scalar triple product to show that the vectors $\vec{a} = \langle 1,4,-7\rangle$, $\vec{b} = \langle 2,-1,4\rangle$, and $\vec{c} = \langle 0,-9,18\rangle$ are Coplanar.

Coplanar: This means all the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are in the same plane and the triple product of them is zero.

solution: $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} + (-7)\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$

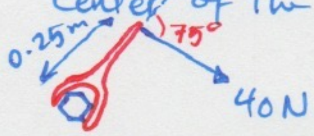
$$\Rightarrow \quad \vec{a}\cdot(\vec{b}\times\vec{c}) = 1(-18-(-36)) - 4(36-0) - 7(-18-0)$$
$$= 18 - 144 + 126 = 0$$

This means the volume of the parallelepiped is zero and therefore all the vectors are in the same plane.

Torque: the idea of cross product occurs often in physics. In particular when a Force vector is acting on a rigid body at a point given by a position vector $\vec{r}$. This is called torque in physics and is defined to be the cross product of the position and the force vectors. Torque is a measure of the tendency of the body to rotate about the origin.

$$\vec{\tau} = \vec{r}\times\vec{F}$$

Example: A bold is tightened by applying a 40 N force to a 0.25 m wrench as shown in the figure below. Find the magnitude of the torque about the center of the bolt. (The angle b/w the force and the wrench handle is 75°)



$$|\tau| = |\vec{r}\times\vec{F}| = |\vec{r}||\vec{F}|\sin 75^\circ = (0.25)(40)\sin 75^\circ$$
$$\approx 9.66 \text{ N.m}$$