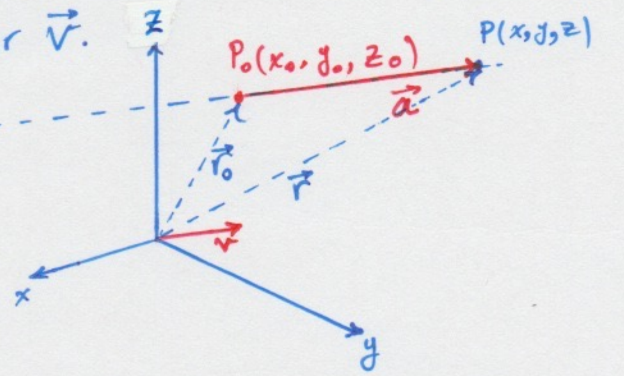
Problem statement: suppose That you are given a point in space $P_0(x_0, y_0, z_0)$ and a direction (say vector $\vec{v}$). You are asked to find the equation of The line ℓ such that it passes through $P_0(x_0, y_0, z_0)$ and is parallel to vector $\vec{v}$.



In order to do this, we pick an arbitrary point on line ℓ and we call that $P(x, y, z)$ and we name $\overrightarrow{P_0P}$ as vector $\vec{a}$ (Notice that vector $\vec{a}$ is parallel to vector $\vec{v}$)

Then we draw the position vectors to P_0 and P and we call Them $\vec{r}_0$ and $\vec{r}$, respectively.

we see from The above figure:

$\vec{r}_0 + \vec{a} = \vec{r}$ → vector Equation

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, $V = \langle a, b, c \rangle$

$\vec{r} = \langle x, y, z \rangle$

We also know That $\vec{a}$ and $\vec{v}$ are parallel but Their magnitude is different

$$\Rightarrow \quad \vec{a} = t\vec{v}$$

we can conclude that $\vec{r}_0 + t\vec{v} = \vec{r}$ which means

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$, in other words $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$

Parametric Equation

These equations are called parametric equations of line ℓ through (x_0, y_0, z_0) and parallel to $\vec{v} = \langle a, b, c \rangle$

* Parametric equations for the line That passes through The point (x_0, y_0, z_0) and parallel to The direction vector $\langle a, b, c \rangle$ are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

Example: find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and Parallel to vector $i+4j-2k$.

$\vec{r}_0 = \langle 5,1,3\rangle$

$\vec{v} = \langle 1,4,-2\rangle$

$\Rightarrow$ (1) $\vec{r} = \vec{r}_0 + t\vec{v}$

$\vec{r} = (5i+j+3k)+t(i+4j-2k) \Rightarrow$

$\vec{r} = (t+5)\,i + (4t+1)\,j + (-2t+3)\,k$: Vector equation

(2) $\langle x,y,z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a,b,c\rangle$

$$\left.\begin{aligned} x &= 5+t \\ y &= 1+4t \\ z &= 3-2t \end{aligned}\right\} : \text{Parametric equations}$$

Example: Find two other points on the line from the previous example.

solution: we can pick any value for t and plug it in the parametric eqs.

$t=1 \longrightarrow x=6,\ y=5,\ z=1 \longrightarrow (6,5,1)$

$t=-1 \longrightarrow x=4,\ y=-3,\ z=5 \longrightarrow (4,-3,5)$

* Note: The vector equation or parametric equation of a line is Not unique. we can pick a different point on the line or another vector parallel to the line and get a different parametric equation.

* Note: Another way to describe a line, ℓ, is to eliminate the parameter t from the parametric equation of a line

$t = \dfrac{x-x_0}{a}$, $t=\dfrac{y-y_0}{b}$, $t=\dfrac{z-z_0}{c}$ $\Rightarrow$ $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

These equations are called Symmetric equations of line ℓ.

Note: If one of the components of vector $V=\langle a,b,c\rangle$ that is Parallel to line ℓ is zero, (meaning $a=0$, or $b=0$ or $c=0$), we can still eliminate t

e.g. if $a=0 \Rightarrow x=x_0$, $\dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$ (All This means is that ℓ lies in the vertical plane $x=x_0$)

Example: Find the Parametric equations and symmetric equations of the line that passes through the points $A(2,4,-3)$ and $B(3,-1,1)$. At what point does this line intersect the xy-plane.

Solution: In this problem instead of one direction vector and one point on the line, we have only two points on the line. we can find vector $\vec{AB}$ and use it as the direction vector for line l

$$\vec{AB} = \langle 3-2, -1-4, 1-(-3) \rangle = \langle 1,-5,4 \rangle \Rightarrow \vec{V} = \langle 1,-5,4 \rangle$$

$$\Rightarrow \left.\begin{array}{l} x = x_0 + ta \Rightarrow x = 2+t \\ y = y_0 + tb \Rightarrow y = 4-5t \\ z = z_0 + tc \Rightarrow z = -3+4t \end{array}\right\} \text{Parametric Equations}$$

$$x-2 = \frac{y-4}{-5} = \frac{z+3}{4} \quad \text{symmetric Equations}$$

we know that all the points on xy-plane have $z=0$

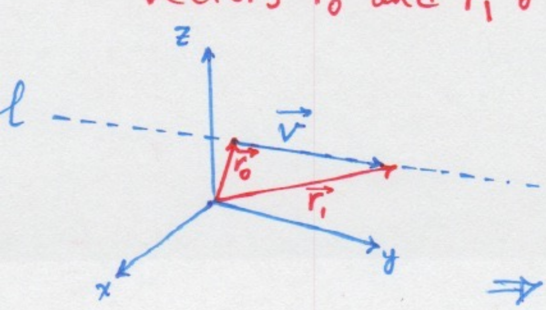
$$x-2 = \frac{0+3}{4} \Rightarrow x = 2+\frac{3}{4} = \frac{11}{4} \Rightarrow x = \frac{11}{4}, \quad \frac{y-4}{-5} = \frac{3}{4} \Rightarrow y = 4 + \frac{(-15)}{4} = \frac{1}{4}$$

The Coordinates of the point the this line intersects with xy-plane at, is $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$

Note: we often need to describe a segment of a line (by limiting t for example) Not the entire line itself.

For example $r(t) = \langle 2+t, 4-5t, -3+4t \rangle$, $0 \le t \le 1$ is the vector equation of a line segment.

The vector equation of a line segment that passes through the tip of position vectors $\vec{r_0}$ and $\vec{r_1}$:



suppose that line L passes through the tip of $\vec{r_0}$ and $\vec{r_1}$

we see that $\vec{r_0} + \vec{V} = \vec{r_1} \Rightarrow \vec{V} = \vec{r_1} - \vec{r_0}$

we also know $\vec{r} = \vec{r_0} + t\vec{v}$

$$\Rightarrow \vec{r} = \vec{r_0} + t(\vec{r_1} - \vec{r_0}) \Rightarrow \vec{r} = (1-t)\vec{r_0} + t\vec{r_1}$$

$$\Rightarrow \vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}, \quad 0 \le t \le 1$$

Example: show that the lines L_1 and L_2 with parametric equations

$$L_1: x = 1+t \ , \ y = -2+3t \ , \ z = 4-t$$

$$L_2: x = 2s \ , \ y = 3+s \ , \ z = -3+4s$$

are skew lines; that is, they do not intersect and are not parallel.

Solution: these line are not parallel because the corresponding direction vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel (their components are not proportional)

If L_1 and L_2 have intersection points, then there should be a $\underline{t}$ and $\underline{\underline{s}}$ value that satisfy these three equations simultaneously

$$\begin{cases} 1+t = 2s & \Rightarrow s = \dfrac{t+1}{2} \\ -2+3t = 3+s & \Rightarrow s = 3t-5 \\ 4-t = -3+4s & \end{cases} \Rightarrow 3t-5 = \frac{t+1}{2} \Rightarrow 6t-10 = t+1$$

$$\Rightarrow 5t = 11 \Rightarrow t = \frac{11}{5}$$

$$\Rightarrow s = \frac{8}{5}$$

Now lets see if these values satisfy the third equation or Not!

$$4-t = 4-\frac{11}{5} = \frac{20-11}{5} = \frac{9}{5}$$

$$-3+4s = -3+(4)\left(\frac{8}{5}\right) = \frac{-15+32}{5} = \frac{17}{5}$$

$$\frac{17}{5} \neq \frac{9}{5}$$

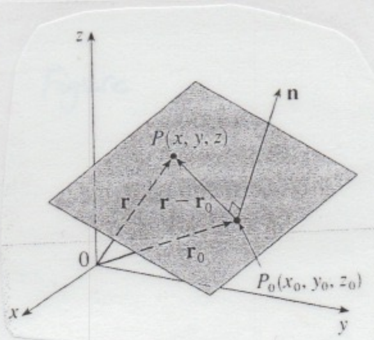
This means the values that we have found for s and t does not satisfy the third equation and as a result, there is no intersection points between L_1 and L_2.

Planes: In the previous section we saw that in order to draw a line in space we need a point like $P_0(x_0, y_0, z_0)$ and a direction.

Similarly, to identify a plane in space we need a point on the plane and a vector (for direction) but this vector should be perpendicular to the plane. We call this perpendicular vector, the **normal vector**.

Suppose that we want to find the equation of a plane that passes through the point $P_0(x_0, y_0, z_0)$ and is perpendicular to vector $\vec{n} = \langle a, b, c\rangle$.

Let $P(x, y, z)$ be an arbitrary point on the plane.



from the figure we can see:

$$r - r_0 = \overrightarrow{P_0P}$$

and $\overrightarrow{P_0P}$ is perpendicular to $\vec{n}$

$$\Rightarrow \vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \text{(vector Equation of the plane)}$$

$$\vec{r} = \langle x, y, z\rangle \ , \ \vec{r_0} = \langle x_0, y_0, z_0\rangle$$

$$\Rightarrow \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

$$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

scalar Equation of the plane

A scalar equation of the plane through point $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

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* You **do not** need to be able to derive the above formula but you certainly need to know the formula and be able to use it :)

Example: Find an equation of the plane through the point $(2,4,-1)$ with normal vector $n=\langle 2,3,4\rangle$. Find the intercepts and sketch the plane.

$n=\langle a,b,c\rangle=\langle 2,3,4\rangle$

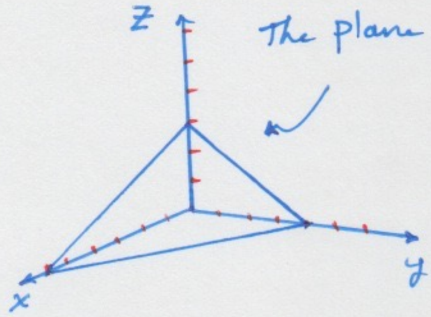
$a(x-x_0)+b(y-y_0)+c(z-z_0)=0 \Rightarrow 2(x-2)+3(y-4)+4(z+1)=0$

$\Rightarrow 2x+3y+4z-4-12+4=0 \Rightarrow \boxed{2x+3y+4z=12}$

x-intercept $(y=0, z=0) \Rightarrow 2x=12 \Rightarrow x=6$

y-intercept $(x=0, z=0) \Rightarrow 3y=12 \Rightarrow y=4$

z-intercept $(y=0, x=0) \Rightarrow 4z=12 \Rightarrow z=3$



Example: Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$.

If we find the vectors $\vec{PQ}$ and $\vec{PR}$ and find their cross product, it will give us a vector that is perpendicular to both these vectors.

$\vec{n}=\vec{PQ}\times\vec{PR}$, $\vec{PQ}=\langle 3-1,-1-3,6-2\rangle=\langle 2,-4,4\rangle$

$\vec{PR}=\langle 5-1,2-3,0-2\rangle=\langle 4,-1,-2\rangle$

$\vec{n}=\vec{PQ}\times\vec{PR}=\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}=12i+20j+14k \rightarrow$ This is the normal vector

with $\vec{n}=\langle 12,20,14\rangle$ and $P(1,3,2)$

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0 \Rightarrow 12(x-1)+20(y-3)+14(z-2)=0$

$\Rightarrow 12x+20y+14z-12-60-28=0$

$12x+20y+14z=100 \rightarrow 6x+10y+7z=50$

Example: Find the point at which the line with parametric equation $x = 2+3t$, $y = -4t$, $z = 5+t$ intersects the plane $4x+5y-2z=18$.

We can easily replace the parametric values for $x, y,$ and z in the equation of plane and solve for t.

$$4(2+3t)+5(-4t)-2(5+t)=18 \Rightarrow 8+12t-20t-10-2t=18$$

$$-10t = 20 \Rightarrow \boxed{t=-2}$$

This means, the intersection point is associated w/ parameter t being equal to -2.

$$\Rightarrow \quad x = 2+3(-2) = 2-6 = -4$$

$$y = -4(-2) = 8$$

$$z = 5+(-2) = 3$$

Point of intersection $(-4, 8, 3)$

Note: Two planes are parallel if their normal vectors are parallel.

e.g. $x+2y-3z=4$ and $2x+4y-6z=3$ are parallel because $\langle 1,2,-3\rangle$ and $\langle 2,4,-6\rangle$ are parallel.

Note: If two planes are not parallel, then the angle between two planes is defined by the acute (means $\theta < 90°$) angle between their normal vectors.

Example: Find the angle between the planes $x+y+z=1$ and $x-2y+3z=1$.
Find symmetric equations for the line of intersection L of these two planes.

(a) The angle between these two planes is the same as the angle between their normal vectors.

$$\Rightarrow \vec{n}_1 = \langle 1,1,1\rangle \quad \vec{n}_2 = \langle 1,-2,3\rangle \Rightarrow \cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|} = \frac{1-2+3}{\sqrt{1^2+1^2+1^2}\cdot\sqrt{1^2+2^2+3^2}}$$

$$\Rightarrow \cos\theta = \frac{2}{\sqrt{3}\cdot\sqrt{14}} = \frac{2}{\sqrt{42}} \Rightarrow \theta = \operatorname{Arc}\cos\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

(b) To find The symmetric equation of the line of intersection (when two planes intersect each other, the geometry of the intersection is a line) we need to find a point on this line.

We can do this by finding the point that the line passes through the xy-Plane ($z=0$).

$$\left.\begin{array}{l} x+y+z=1,\ z=0 \Rightarrow x+y=1 \\ x-2y+3z=1,\ z=0 \Rightarrow x-2y=1 \end{array}\right\} \text{ we solve this system to find } x \text{ and } y \quad x=1,\ y=0$$

Point $(1,0,0)$ is the coordinates of the intersection of the line L w/ xy-plane.

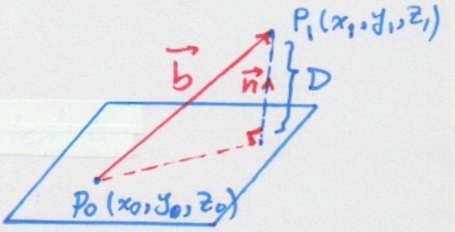
We know that line L, lies in both planes, therefore, it is perpendicular to both of their normal vectors. This means by calculating the cross product of $\vec{n}_1 \times \vec{n}_2$ we can find a direction that is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$ and That would be the direction of the line ℓ.

$$\vec{n}_1\cdot\vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5i-2j-3k$$

(steps of calculation of the cross product have been eliminated)

Therefore the equation of a line that passes through $(1,0,0)$ and has the direction vector of $\langle 5,-2,-3\rangle$ is $\frac{x-1}{5} = \frac{y-0}{-2} = \frac{z-0}{-3}$

Distances: Find a formula for the distance D, from a point $P_1(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$. (Problem statement)



Let $P_0(x_0, y_0, z_0)$ be any point in the given plane.

$\overrightarrow{P_0P_1} = \langle x_1-x_0, y_1-y_0, z_1-z_0 \rangle = \vec{b}$ (for simplicity we call it $\vec{b}$)

from the figure, we can see that the distance D from P_1 to the plane is the same as $\text{Comp}_{\vec{n}}\, b$. $\vec{n} = \langle a, b, c \rangle$

$$D = |\text{Comp}_n \vec{b}| = \frac{|\vec{n}\cdot\vec{b}|}{|n|} = \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow D = \frac{|(ax_1+by_1+cz_1)-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}}$$

Since P_0 is on the plane and any point on the plane satisfy $ax+by+cz+d=0$

$\Rightarrow ax_0+by_0+cz_0 = -d$

$$\Rightarrow \boxed{D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}} \text{ (Important)}$$

Example: Find the distance between the parallel planes $10x+2y-2z=5$ and $5x+y-z=1$.

It is obvious that these two planes are parallel because $\vec{n}_1 \parallel \vec{n}_2$ (Their normal vectors are parallel). $\vec{n}_1 = \langle 10, 2, -2 \rangle$, $\vec{n}_2 = \langle 5, 1, -1 \rangle$

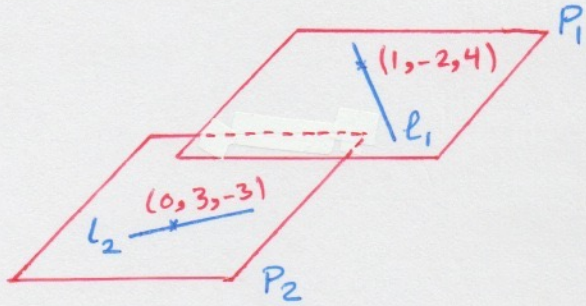
To find the distance between two planes, we find one point on a plane and we calculate its distance to the other plane.

If we set $y=z=0$ on plane one we get $10x+2(0)-2(0)=5$

$\Rightarrow x = \frac{1}{2}$ and $(\frac{1}{2}, 0, 0)$ is a point on that plane (This is an arbitrary point and we could set y and z to anything else!

$$\Rightarrow D = \frac{|5(\frac{1}{2})+1(0)+1(0)-1|}{\sqrt{5^2+1^2+1^2}} = \frac{3}{2\sqrt{27}} = \frac{3}{2\cdot 3\sqrt{3}} = \frac{\sqrt{3}}{6} \Rightarrow D = \frac{\sqrt{3}}{6}$$

Example: In one of our previous examples we showed that the following parametric lines are skew.



$L_1: x = 1+t, \; y = -2+3t, \; z = 4-t$

$L_2: x = 2s, \; y = 3+s, \; z = -3+4s$

Find the distance between them.

L_1 and L_2 are skew so we can assume they both lie on planes P_1 and P_2 that are parallel to each other. Now we need to find a normal vector (perpendicular to any of these planes) and this common normal is orthogonal to both L_1 and L_2:

$\Rightarrow \vec{n} = \vec{V}_1 \times \vec{V}_2$ ($\vec{V}_1$ and $\vec{V}_2$ are the direction vectors of L_1 and L_2)

$$\Rightarrow \vec{n} = \begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13i - 6j - 5k$$ (This is the common normal vector)

Now we need to find a point on P_2 plane (we could do it with P_1 plane as well) so we can write an equation for plane P_2.

We know that L_2 lies in P_2 so we set $s=0$ and find coordinates of a point on line L_2. $s=0 \Rightarrow x=0, y=3, z=-3$ $(0,3,-3)$

Equation of plane P_2: $13(x-0) + (-6)(y-3) + (-5)(z+3) = 0$

$$\Rightarrow 13x - 6y - 5z = -18 + 15 \Rightarrow \underbrace{\boxed{13x - 6y - 5z + 3 = 0}}_{P_2}$$

Now we find a point on L_1 by setting $t=0 \Rightarrow (1,-2,4)$ on L_1

Now we calculate the distance between the point on L_1 and plane P_2

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$