Section 13.1

From Math 2A, you remember that we defined a function as a rule that assigns to each element in the domain an element in the range.

A Vector Valued Function or a vector function is simply a function whose domain is a set of real numbers and whose range is a set of vectors.

Think about vector function $r(t)$ whose values are three dimensional vectors. Therefore for every number t in the domain, there is a unique vector in V_3 (three dimensional space) denoted by $\vec{r}(t)$.

Assuming $f(t)$, $g(t)$, and $h(t)$ are the components of vector $\vec{r}(t)$, Then f, g, and h are real-valued functions called the component functions of $\vec{r}(t)$ and we can write

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\, i + g(t)\, j + h(t)\, k$$

We normally use the independent variable t because in most vector function applications, it represents time.

Example: If vector function $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$ then find the domain of the function.

Solution: $f(t) = t^3 \Rightarrow$ domain $= \mathbb{R}$ (All the real numbers)

$g(t) = \ln(3-t) \Rightarrow 3-t > 0 \Rightarrow t < 3$ domain: $t < 3$

$h(t) = \sqrt{t} \Rightarrow t \geq 0 \Rightarrow$ domain: $t \geq 0$

$\Rightarrow$ Domain of the vector function $\vec{r}(t)$ is $[0, 3)$

limits & Continuity:

The limit of a vector function $\vec{r}$ is defined by taking the limits of its component functions.

* If $r(t) = \langle f(t), g(t), h(t) \rangle$ then $\lim\limits_{t \to a} \vec{r}(t) = \langle \lim\limits_{t \to a} f(t), \lim\limits_{t \to a} g(t), \lim\limits_{t \to a} h(t) \rangle$ provided the limits of the component functions exist.

A vector function is continuous at a if $\lim\limits_{t \to a} \vec{r}(t) = \vec{r}(a)$

Example: find $\lim\limits_{t \to 0} \vec{r}(t)$, where $r(t) = (1+t^3)i + te^{-t}j + \frac{\sin t}{t} k$.

$$\lim_{t \to 0} \vec{r}(t) = [\lim_{t \to 0} (1+t^3)]i + [\lim_{t \to 0} te^{-t}]j + [\lim_{t \to 0} \frac{\sin t}{t}]k$$

$\lim\limits_{t \to 0} 1+t^3 = \boxed{1}$ $\lim\limits_{t \to 0} te^{-t} = 1 \times 0 = \boxed{0}$ $\lim\limits_{t \to 0} \frac{\sin t}{t} = \frac{0}{0}$ L'Hospital

$\lim\limits_{t \to 0} \frac{\cos t}{1} = \boxed{1}$

$\Rightarrow \lim\limits_{t \to 0} \vec{r}(t) = i + k$

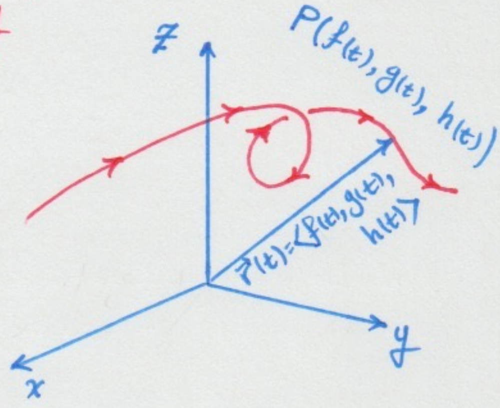
Space Curves:

There is a close connection between continuous vector functions and space curves. suppose that f, g, and h are continuous real-valued functions on interval I. Then, the set C of all points in space (x, y, z), where $x = f(t)$, $y = g(t)$, $z = h(t)$ for all values of t on interval I is called a Space Curve.

If we consider the vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\vec{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on space curve C.

* Therefore, any continuous vector function $\vec{r}(t)$ defines a space curve C that is traced out by the tip of the vector $\vec{r}(t)$.

* In simple words, components of a vector function (They are also components of the position vector) trace the space curve that is the set of all (x, y, z) points such that
  $x = f(t)$ $\quad y = g(t)$ $\quad z = h(t)$



Example: Describe the curve defined by the vector function $\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$.

Solution: the parametric equation $x = 1+t$, $y = 2+5t$ and $z = -1+6t$ represents the parametric equation of line that passes through the points $(+1, +2, -1)$ and is parallel to the vector $\langle 1, 5, 6 \rangle$

Example: sketch the curve whose vector equation is
$r(t) = \cos t\, i + \sin t\, j + t\, k$.

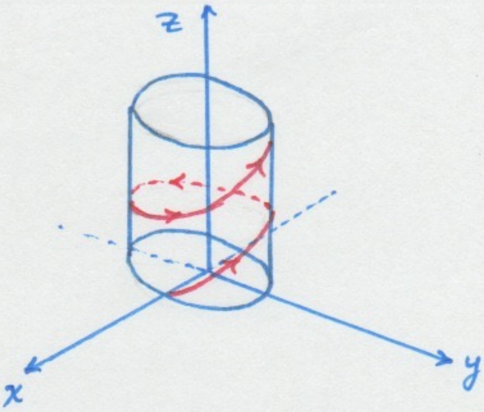
Solution: the parametric equations for the curve is
$x = \cos t$, $y = \sin t$, $z = t$

since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, then for all values of t, the curve must lie on the circular cylinder (Pay attention! a cylinder not a circle) w/ the equation $x^2 + y^2 = 1$

Continued in the next page.

* it is important to note that the projection of this space curve has a vector equation as $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$. Since $z = t$, the curve spirals upward around the cylinder as t increases.

projection on xy-Plane



Example: Find a vector equation and parametric equations for the line segment that joins the point $P(1,3,-2)$ to the point $Q(2,-1,3)$.

Solution: from section 12.5 we remember the equation of a line segment that join the tip of vectors $\vec{r}_0$ and $\vec{r}_1$ is

$$r(t) = (1-t)r_0 + tr_1 \qquad 0 \leq t \leq 1$$

$$r_0 = \langle 1, 3, -2 \rangle \quad , \quad r_1 = \langle 2, -1, 3 \rangle$$

$$\Rightarrow \quad r(t) = (1-t)\langle 1,3,-2 \rangle + t\langle 2,-1,3 \rangle$$

$$\Rightarrow \quad r(t) = \langle 1-t+2t \, , \, 3-3t-t \, , \, -2+2t+3t \rangle$$

$$r(t) = \langle 1+t \, , \, 3-4t \, , \, -2+5t \rangle$$

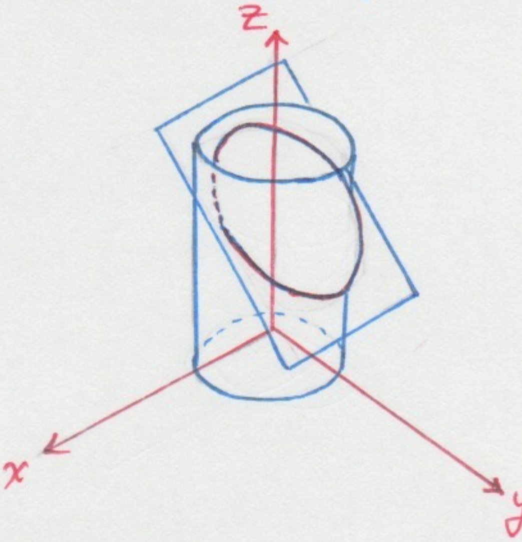
The corresponding parametric equations are

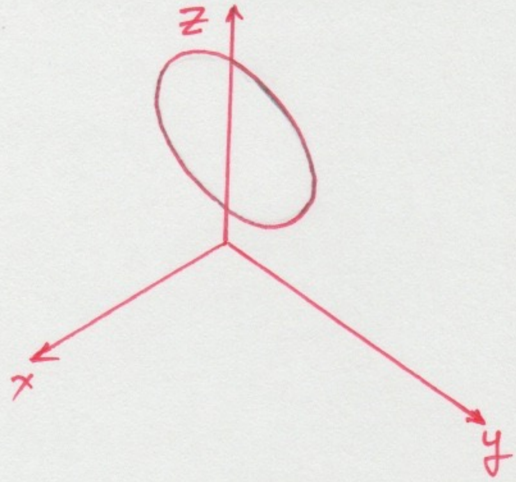
$$x = 1+t$$

$$y = 3-4t$$

$$z = -2+5t$$

Example: Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $y+z=2$.





The projection of C onto the xy-plane is the circle with $x^2+y^2=1$, $z=0$.

$\Rightarrow$ $x=\cos t$, $y=\sin t$, and $0\leq t\leq 2\pi$

from the equation of the plane we have:

$$z=2-y=2-\sin t$$

Therefore, the parametric equation for curve C is:

$$r(t)=\cos t\, i+\sin t\, j+(2-\sin t)\, k \quad 0\leq t\leq 2\pi$$

$\hookrightarrow$ This process is called parametrization of curve C.