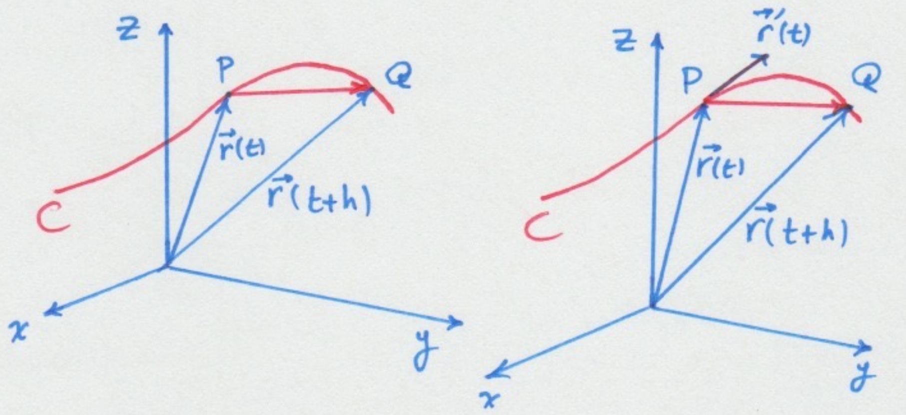
The derivative $\vec{r}'(t)$ of a vector function is defined in the same way as for a real valued function:

$$\boxed{\frac{d\vec{r}(t)}{dt} = r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}}$$ (if the limit exists)

The geometric significance of $\vec{r}(t)$ can be seen in the following figures:



If we assume that $\vec{r}(t)$ is the position vector of space curve C for t and $\vec{r}(t+h)$ is the position vector associated w/ $(t+h)$, then $\vec{PQ} = \vec{r}(t+h) - \vec{r}(t)$. Now if we make h smaller and smaller, $\vec{PQ}$ vector approaches to $\vec{r}'(t)$. $\vec{r}'(t)$ is called the tangent vector.

Note: $\vec{T}(t) = \dfrac{r'(t)}{|r'(t)|}$ is the unit tangent vector.

Note: If we are interested in finding the equation of a line that is tangent to curve C at point P, we can pick point P as a point on the line and $\vec{r}'(t)$ as the vector that the tangent line is parallel to. (Remember that in order to find the equation of a line in three dimensional space, we need to have a point that lies on the line and a vector that is parallel to the line)

* Theorem: If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,i + g(t)\,j + h(t)\,k$ where f, g, and h are differentiable functions, then

$$r'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\,i + g'(t)\,j + h'(t)\,k$$

Example: (a) Find the derivative of $\vec{r}(t) = (1+t^3)\,i + te^{-t}\,j + \sin 2t\,k$

(b) Find the unit tangent vector at the point where $t=0$

(a) $r'(t) = 3t^2\,i + (e^{-t} - te^{-t})\,j + 2\cos 2t\,k$

(b) $T(0) = \dfrac{r'(0)}{|r'(0)|} = \dfrac{j + 2k}{\sqrt{1^2+2^2}} = \dfrac{1}{\sqrt{5}}\,j + \dfrac{2}{\sqrt{5}}\,k$

Example: for the curve $r(t) = \sqrt{t}\,i + (2-t)\,j$ find $r'(t)$ and sketch the position vector $r(t)$ and the tangent vector $r'(1)$.

$r'(t) = \dfrac{1}{2\sqrt{t}}\,i - j \quad , \quad r'(1) = \dfrac{i}{2} - j$

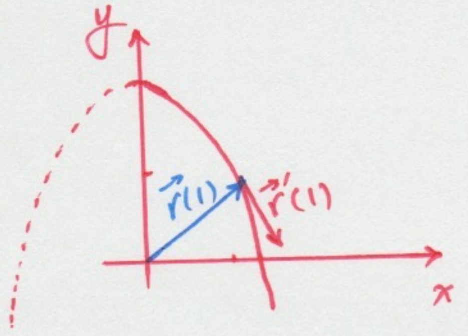
$r(t) = \sqrt{t}\,i + (2-t)\,j \Rightarrow \begin{array}{l} x(t) = \sqrt{t} \Rightarrow t = x^2 \\ y(t) = 2-t \Rightarrow t = 2-y \end{array} \Big\} \Rightarrow x^2 = 2-y$

$\Rightarrow y = -x^2 + 2$

This is a plane curve in xy plane and is a parabola.

$x \geqslant 0$

$\Rightarrow$



$t=1 \rightarrow x = 1$

$y = 2-1 = 1$

Example: Find Parametric equations for the tangent line to the helix with parametric equations:

$x = 2\cos t \quad y = \sin t \quad z = t$

at the point $(0, 1, \frac{\pi}{2})$.

$t = \frac{\pi}{2}$, $\sin\frac{\pi}{2} = 1$, $2\cos\frac{\pi}{2} = 0$

$\Rightarrow t = \frac{\pi}{2}$ is associated w/ the given point.

$r'(t) = \langle -2\sin t, \cos t, 1 \rangle \Rightarrow r'(\frac{\pi}{2}) = \langle -2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1 \rangle$

$\Rightarrow r'(\frac{\pi}{2}) = \langle -2, 0, 1 \rangle \Rightarrow$ we have a point on the line and we have a vector that is parallel to this parallel line.

The parametric eq. of the line is:

$x = -2t$, $y = 1$, $z = \frac{\pi}{2} + t$

Differentiation Rules for Vector Valued Functions:

Suppose that $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector functions, c is a scalar, and f is a real valued function:

$$\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$$

$$\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$$

$$\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

Dot product

$$\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

$$\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

Similar to the Product Rule.

$$\frac{d}{dt}[u(f(t))] = f'(t)\,u'(f(t))$$

Chain Rule

Example: Show that if $|\vec{r}(t)| = c$ (a constant), then $r'(t)$ is orthogonal to $r(t)$ for all t.

we know that for every vector $\vec{a}.\vec{a} = |a|^2$ (Because $\vec{a}.\vec{a} = |a||a|.\cos\theta$ and $\theta = 0 \Rightarrow \cos\theta = 1$)

$\Rightarrow \vec{r}(t).\vec{r}(t) = |r(t)|^2 = c^2$

we take the derivative w/ respect to t from Both sides and we will have

$$\frac{d}{dt}\left(\vec{r}(t).\vec{r}(t)\right) = \underbrace{\frac{d}{dt}(c^2)}_{\emptyset} \Rightarrow \vec{r}'(t).\vec{r}(t) + \vec{r}(t).\vec{r}'(t) = 0$$

$\Rightarrow \vec{r}'(t).\vec{r}(t) = \emptyset \Rightarrow$ This means since the dot product of Two vectors is zero, they must by orthogonal.

Integrals of vector valued Functions:

The definite integral of a continuous vector function $\vec{r}(t)$ can be defined in much the same way as for real valued functions except that the integral is a vector.

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\underline{\underline{i}} + \left(\int_a^b g(t)\,dt\right)\underline{\underline{j}} + \left(\int_a^b h(t)\,dt\right)\underline{\underline{k}}$$

Pay attention!

* Note: We can extend the fundamental theorem of calculus to continuous vector functions as follows:

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big]_a^b = \vec{R}(b) - \vec{R}(a)$$

where $\vec{R}$ is the antiderivative of r, that is, $R'(t) = r(t)$

* Note: we use the notation $\int \vec{r}(t)\, dt$ for indefinite integrals (Antiderivatives)

Example: If $r(t) = 2\cos t\, i + \sin t\, j + 2t\, k$, Then find $R(t)$, The antiderivative of $r(t)$, and then calculate $\int_0^{\frac{\pi}{2}} r(t)\, dt$.

$$\int \vec{r}(t)\, dt = \left(\int 2\cos t\, dt\right) i + \left(\int \sin t\, dt\right) j + \left(\int 2t\, dt\right) k$$

$$\vec{R}(t) = 2\sin t\, i - \cos t\, j + t^2 k + \vec{C}$$

This is a vector!
we call it vector constant of integration!

$$\int_0^{\frac{\pi}{2}} r(t) = 2\sin t\, i - \cos t\, j + t^2 k \Big|_0^{\frac{\pi}{2}}$$

$$= 2\sin\frac{\pi}{2}\, i - \cos\frac{\pi}{2}\, j + \left(\frac{\pi}{2}\right)^2 k - \left(2\sin 0\, i - \cos 0\, j + (0)^2 k\right)$$

$$= 2i + \frac{\pi^2}{4} k + j = 2i + j + \frac{\pi^2}{4} k$$