In chapter 10 we defined the length of a plane curve with parametric equations $x=f(t)$, $y=g(t)$, $a\leq t\leq b$ as the following:

$$\ell=\int_a^b\sqrt{[f'(t)]^2+[g'(t)]^2}\cdot dt=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

The length of the space curves (in three dimensions) is defined in the same way.

Suppose that the curve has the vector equation $r(t)=\langle f(t), g(t), h(t)\rangle$, $a\leq t\leq b$. In other words, $x=f(t)$, $y=g(t)$, $z=h(t)$.

If the curve is transversed exactly once as t increases from a to b:

$$\boxed{\ell=\int_a^b\sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}\cdot dt}$$

$$\boxed{\ell=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\cdot dt}$$

If we assume $r(t)=\langle f(t), g(t), h(t)\rangle \Rightarrow r'(t)=\langle f'(t), g'(t), h'(t)\rangle$.

$$\Rightarrow |r'(t)|=\sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}$$

$$\Rightarrow \boxed{\ell=\int_a^b|r'(t)|\,dt}$$

Example: Find the length of the Arc of the circular helix with vector equation $r(t)=\cos t\, i+\sin t\, j+t\, k$ from the Point $(1,0,0)$ to the point $(1,0,2\pi)$.

Solution: $r'(t)=-\sin t\, i+\cos t\, j+k$

$$\Rightarrow |r'(t)|=\sqrt{(-\sin t)^2+(\cos t)^2+(1)^2}=\sqrt{2}$$

$$\ell=\int_0^{2\pi}\sqrt{2}\,dt=\sqrt{2}\,t\Big|_0^{2\pi}=\boxed{2\pi\sqrt{2}}$$

Point $(1,0,0)$

$$\begin{cases} x:\ \cos t=1\Rightarrow t=0,\ 2\pi\\ y:\ \sin t=0\Rightarrow t=0,\ \pi,\ 2\pi\\ z:\ t=0\end{cases}$$

Point $(1,0,2\pi)$

$$\begin{cases} x:\ \cos t=1\Rightarrow t=0,\ 2\pi\\ y:\ \sin t=0\Rightarrow t=0,\ \pi,\ 2\pi\\ z:\ t=2\pi\end{cases}$$

* Important Note: A single curve C can be represented by more than one vector function. For instance:

$r_1(t) = \langle t, t^2, t^3 \rangle \quad , 1 \leq t \leq 2$

could also be represented by the function

$r_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle, \; 0 \leq u \leq \ln 2$

In this case the connection b/w t and u is $t = e^u$. These are all different parametrization of the same curve.

The important thing to note is that regardless of the paramtrization eqs. of the curve, the arc length will be equal/the same.

The Arc Length Function

From Math 2B you remember that we defined a function called the Arc Length Function:

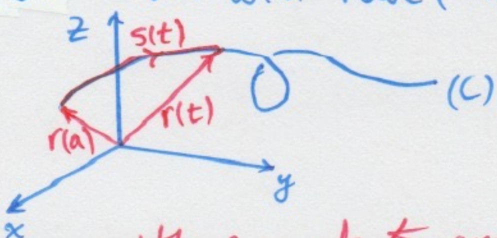
$S(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$, this function gives us the Arc length from point a to any arbitrary point x along x-axis.

In a similar way we can define the Arc Length Functions for space curves:

$$S(t) = \int_a^t |r'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

This means that $S(t)$ is the length of the part of C between $r(a)$ and $r(t)$.

If we differentiate both sides of the equation we will have (Fundamental Theorem of calculus part I)



$$\frac{ds}{dt} = \frac{d}{dt} \int_a^t |r'(u)|\, du = |r'(t)|$$

This is called parametrization of a curve with respect to arc length.

Example: Reparametrize the helix $r(t) = \cos t\, i + \sin t\, j + t\, k$ with respect to arc length along the curve from its starting point (1,0,0) in the direction of increasing t.

The initial point is associated w/ $t=0$.

$$\frac{ds}{dt} = |r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = \sqrt{2}$$

$$s(t) = \int_0^t |r'(u)|\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t \Rightarrow s(t) = \sqrt{2}t \Rightarrow t = \frac{s}{\sqrt{2}}$$

$$\Rightarrow r(t(s)) = \cos\left(\frac{s}{\sqrt{2}}\right) i + \sin\left(\frac{s}{\sqrt{2}}\right) j + \left(\frac{s}{\sqrt{2}}\right) k$$

Normal & Binormal Vectors

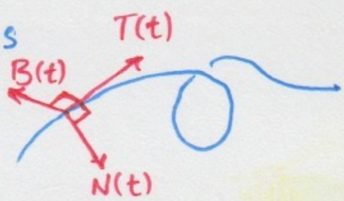
We know at any given point on a space curve, $r(t)$, there are many vectors (infinite) that are perpendicular to the unit Tangent vector $T(t)$. We single out one of these vectors. We can claim that $T'(t)$ is perpendicular to $T(t)$. (why, remember that $|T(t)| = 1$, a constant and refer to the example that we solved in the previous section).

In general $|T'(t)| \neq 1$ and $T'(t)$ is not a unit vector.

We define Principal unit normal vector as $N(t) = \dfrac{T'(t)}{|T'(t)|}$

We can think of $\vec{N}(t)$ as indicating the direction in which the curve is turning at each point.

As we can see in this figure, there is another vector that is perpendicular to both $T(t)$ and $T'(t)$ (N(t)). This vector is called Binormal vector and we show it as $B(t) = T(t) \times N(t)$



Notice that $\vec{B}(t)$ is also a unit vector.

Example: Find the unit normal and binormal vectors for the circular helix

$$r(t) = \cos t\, i + \sin t\, j + t k$$

$$r'(t) = -\sin t\, i + \cos t\, j + k \quad , \quad |r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$

$$\Rightarrow \quad \Rightarrow T(t) = \frac{-\sin t\, i + \cos t\, j + k}{\sqrt{2}}$$

$$|T'(t)| = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} = \sqrt{\frac{1}{2}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} \Rightarrow T'(t) = \frac{-\cos t\, i - \sin t\, j}{\sqrt{2}} \quad , \quad |T'(t)| = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow N(t) = \sqrt{2}\left(\frac{-\cos t\, i - \sin t\, j}{\sqrt{2}}\right) = -\cos t\, i - \sin t\, j$$

$$B(t) = T(t) \times N(t) \Rightarrow B(t) = \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} \times \frac{1}{\sqrt{2}}$$

$$\Rightarrow B(t) = \frac{1}{\sqrt{2}}\left[i(0 + \sin t) - j(0 + \cos t) + k(\sin^2 t + \cos^2 t)\right]$$

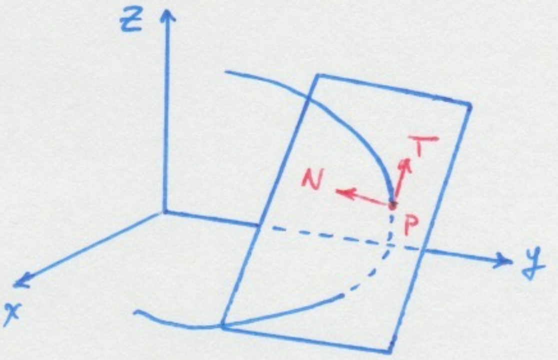
$$B(t) = \frac{1}{\sqrt{2}}(\sin t\, i - \cos t\, j + k)$$

Definition of Normal plane and osculating plane

Normal plane: The plane that is determined by the normal and binormal vectors ($\vec{N}$ and $\vec{B}$) at point P of curve c, is called the normal plane at point P. The normal plane consists of all lines that are orthogonal to the Tangent vector $T(t)$.

Osculating plane: The plane that is determined by the vectors T and N, at point P of the space curve c.
The osculating plane is the plane that comes the closest to containing the part of the curve near point P.

This is how the osculating plane look in three dimensional space:



Example: Find the equations of the normal plane and osculating plane of the helix in the previous example at $P(0,1,\frac{\pi}{2})$.

Solution: The point $(0,1,\frac{\pi}{2})$ corresponds to $t=\frac{\pi}{2}$

$r(t)=\cos t\, i+\sin t\, j+t\, k$

The direction of the normal vector of the normal plane is along T, or $r'(t)$. (Do not confuse this normal vector of the plane with $\vec{N}(t)$).

$\Rightarrow r'(t)=-\sin t\, i+\cos t\, j+k \Rightarrow r'(\frac{\pi}{2})=\langle -1,0,1\rangle$

$\Rightarrow$ The equation of the normal plane is : $-1(x-0)+0(y-1)+1(z-\frac{\pi}{2})=0$

$$\Rightarrow z=x+\frac{\pi}{2}$$

The normal vector of the osculating plane is parallel to $\vec{B}(t) \Rightarrow$

$B=T\times N \Rightarrow$ using the cross product of $T\times N$ we find

$\vec{B}(t)=\frac{1}{\sqrt{2}}\langle \sin t,-\cos t,1\rangle$ and $\vec{B}(\frac{\pi}{2})=\langle\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\rangle$ which is parallel to $\langle 1,0,1\rangle$ → it is just simpler to work w/ this vector

$\Rightarrow$ the equation of the osculating plane : $1(x-0)+0(y-1)+1(z-\frac{\pi}{2})=0$

$$\Rightarrow z=-x+\frac{\pi}{2}$$