section 14.1

A <u>function of two variables</u> is a rule that assigns to each ordered Pair of real numbers (x,y) in a set D a unique real number denoted by $f(x,y)$. The set D is the <u>domain</u> of f and its <u>range</u> is the set of values that f takes on, That is $\{f(x,y) \mid (x,y) \in D\}$ (All The values of $f(x,y)$ such that (x,y) belong to set D)

To understand functions of several variables (Two or more) better, think about The temperature variation in a room. As we get closer to The Heater the temperature of that point increases. so, Temperature is a function of x, y, and possibly z (The temperature of areas closer to The ceiling is higher Than other places). $\Rightarrow$ Temp $= f(x,y,z)$ or in 2D: Temp $= g(x,y)$

x and y Are <u>independent Variables</u>

and $f(x,y)$ (sometimes we replace it with z) is The <u>dependent variable</u>

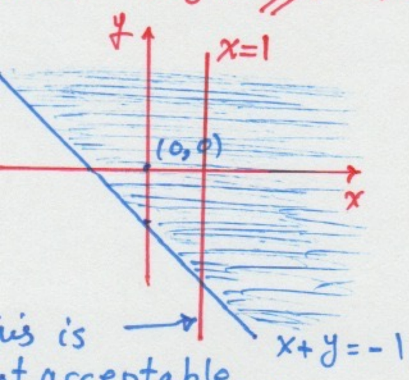
Example: For each of The following functions, evaluate $f(3,2)$ and find and sketch The domain of $f(x,y)$

a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$, $f(3,2) = \dfrac{\sqrt{6}}{2}$

The denominator Can not be zero

$\Rightarrow x \neq 1$

Also, $x+y+1 \geq 0 \Rightarrow x+y \geq -1$



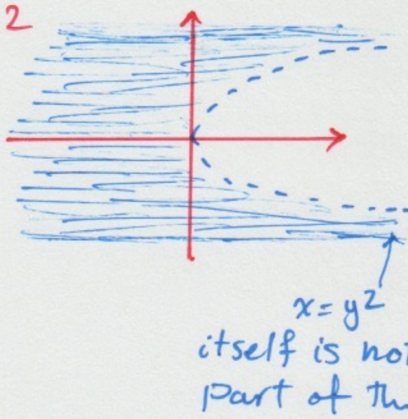
In order to find the half-plane that contains all the points that $x+y \geq -1$ we pick an arbitrary Point and see if it satisfies The inequality $\rightarrow (0,0)$ e.g. (0,0) satisfies The inequality

This is not acceptable as The part of domain

b) $f(x,y) = x \ln(y^2 - x)$ $f(3,2) = 0$

$y^2 - x > 0 \Rightarrow y^2 > x$

we go ahead and sketch $x = y^2$



$x = y^2$ itself is not part of The domain

Example: Find the domain and range of $g(x,y)=\sqrt{9-x^2-y^2}$.

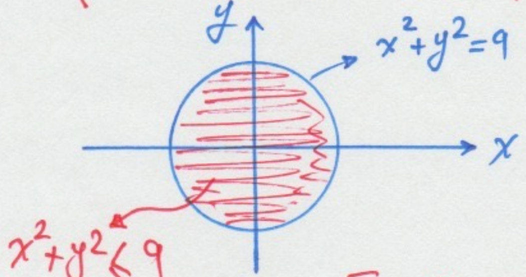
for the domain we know that $9-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 9$

This inequality is the equation of circle that its radius is 3 and is centered at the origin of the coordinate system.

Also, since this is an inequality, we need to know whether the inside or the outside of the circle is included (All the points inside or all the points outside)

$\Rightarrow$ We pick $(0,0)$ as our test point $\rightarrow 0^2+0^2 \leq 9$ (True)

(This means all the points inside the circle are acceptable/included)



So the domain is a disk centered at the origin of the Coordinate system and with the radius of 3.

$D=\{(x,y) \mid x^2+y^2 \leq 9\}$

For range we need to see what are the max and min values that g can take. $g(x,y)=\sqrt{9-x^2+y^2}$

$$\left.\begin{array}{l}\text{Max of } x^2+y^2=9\\ \text{Min of } x^2+y^2=0\end{array}\right\} \Rightarrow \begin{array}{l}\text{Max } g(x,y)=\sqrt{9-0}=3\\ \text{Min } g(x,)=\sqrt{9-9}=0\end{array}$$

$\Rightarrow$ Range $=[0,3]$

$\hookrightarrow$ This means $g(x,y)$ or z is always between 0 and 3.

Graphs: Another way of visualizing the behavior of function of two variables is to consider its graph.

Definition: If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x,y)$ and (x,y) is in D.

Example: sketch the graph of function $f(x,y) = 6 - 3x - 2y$.

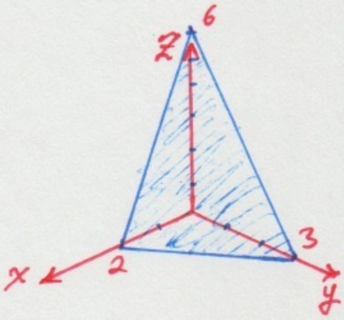
$\Rightarrow z = 6 - 3x - 2y \Rightarrow 3x + 2y + z = 6$ is a plane

to sketch this plane we can find the intersection points of this plane w/ x, y, and z axis.

x-axis intersection $(y = z = 0) \Rightarrow 3x = 6 \Rightarrow \boxed{x = 2}$

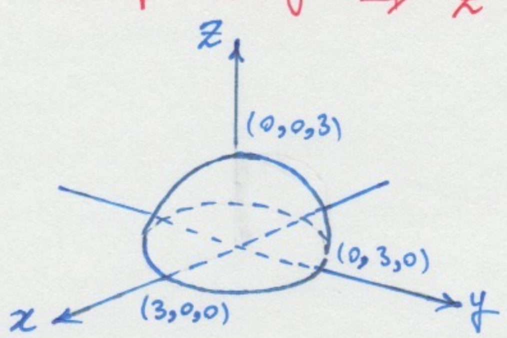
y-axis intersection $(x = z = 0) \Rightarrow 2y = 6 \Rightarrow \boxed{y = 3}$

z-axis intersection $(y = x = 0) \Rightarrow \boxed{z = 6}$



Example: sketch the graph of $g(x,y) = \sqrt{9 - x^2 - y^2}$.

$z = \sqrt{9 - x^2 - y^2} \Rightarrow z^2 = 9 - x^2 - y^2 \Rightarrow x^2 + y^2 + z^2 = 9$ This is the eq. of a sphere with $r = 3$ and centered at the origin of 3D coordinate system. However $x^2 + y^2 + z^2 = 9$ is the eq. that we constructed based on $g(x,y)$.



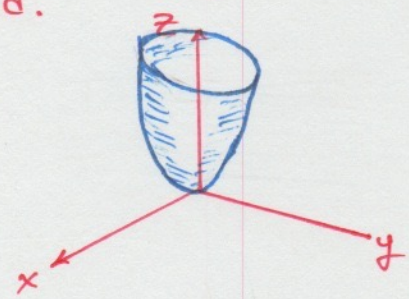
In reality $g(x,y) = \sqrt{9 - x^2 - y^2}$ so $g(x,y)$ or z, can only have positive values. so, $z = \sqrt{9 - x^2 - y^2}$ is the eq. of a hemisphere or half of the sphere that is above the xy-plane

Example: Find the domain and range and sketch the graph of $h(x,y) = 4x^2 + y^2$

$Z = 4x^2 + y^2 \longrightarrow$ Notice that x, and y can take any real numbers.
So the domain is $\mathbb{R}^2$. Since $x^2 \geqslant 0$ and $y^2 \geqslant 0$, Therefore, $Z \geqslant 0$
which means the range of function is $[0, +\infty)$

If we sketch the traces, we realize that the projections of the surface on xy-plane are ellipses ($4x^2 + y^2 = k$), the projections on yz-plane (And all the planes parallel to yz) are parabolas, and the projections of the surface on xz-plane (And all the planes parallel to xz) are parabolas as well. $\left\{ \begin{array}{l} Z = 4x^2 + k \\ Z = k + y^2 \end{array} \right\}$ k is a constant

So the surface is an elliptic paraboloid.



Level Curves:

Another way of visualizing the multivariable functions is to draw a contour map on which points of constant elevation are joined to form contour curves or level curves.

Definition: The level curves of a function f of two variables are the curves with equations $f(x,y) = k$, where k is a constant (in the range of f).

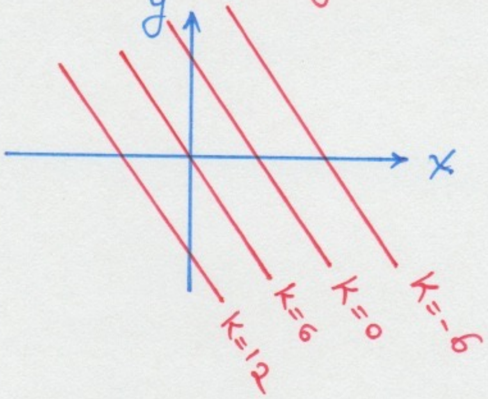
In other words if we sketch the curves of intersection of different planes parallel to xy-plane with the surface (At different heights) on a two dimensional coordinate system, we will have the contour map or the level curves of that surface.

Example: Sketch the level curves of the function $f(x,y) = 6 - 3x - 2y$ for the values $K = -6, 0, 6, 12$.

$z = 6 - 3x - 2y \Rightarrow 3x + 2y + z - 6 = 0$, level curves are created by intersecting $f(x,y)$ w/ planes with eq. $z = k$

$$\Rightarrow 3x + 2y + (K-6) = 0 \longrightarrow \begin{cases} K=-6 \rightarrow 3x+2y-12=0 \\ k=0 \rightarrow 3x+2y-6=0 \\ K=6 \rightarrow 3x+2y=0 \\ K=12 \rightarrow 3x+2y+6=0 \end{cases}$$

These are all the eqs. of level curves in xy-plane



In other words if we sketch the projections of the lines created by the intersection of the plane $3x+2y+z-6=0$ with planes parallel to xy plane with the equations $z=-6$, $z=0$, $z=6$, and $z=12$ we get this contour maps or level curves.

Example: Sketch the level curves of the function $g(x,y) = \sqrt{9-x^2-y^2}$ for $K = 0, 1, 2, 3$.

$z = \sqrt{9-x^2-y^2} \Rightarrow x^2+y^2+z^2 = 9$, $z = k \Rightarrow x^2+y^2 = 9-k^2$

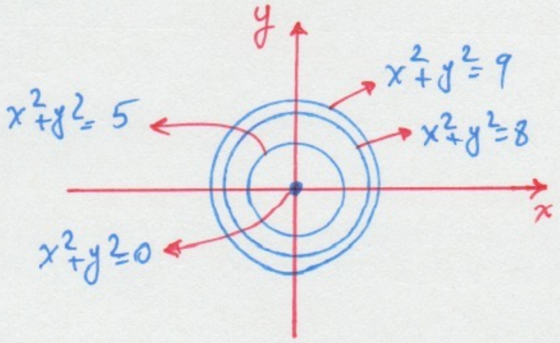
The level curves are the family of concentric circles centered at $(0,0)$ and radius $\sqrt{9-k^2}$.

$x^2+y^2=9$ $K=0$

$x^2+y^2=8$ $K=1$

$x^2+y^2=5$ $K=2$

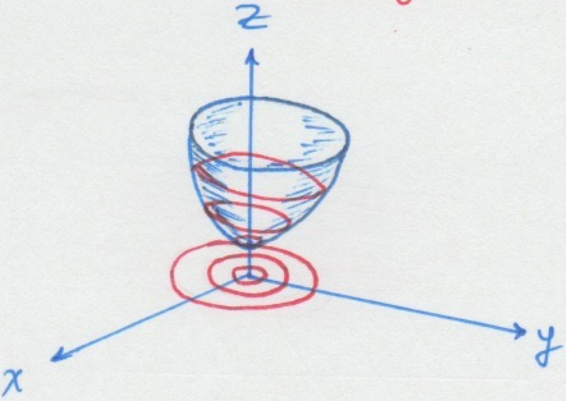
$x^2+y^2=0$ $K=3$

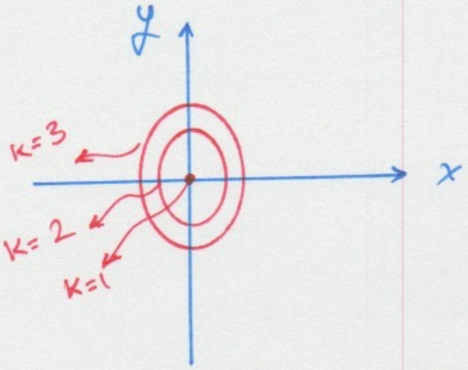


Notice that k values can not be negative because the eq. of $g(x,y)$ represent a hemisphere above xy-plane (z is always positive)

Example: Sketch some level curves of the function $h(x,y)=4x^2+y^2+1$.

$$4x^2+y^2+1=k$$





k cannot be smaller than 1 otherwise $4x^2+y^2<0$ (Not Possible)

Functions of three or more variables:

A function of three variables, f is a rule that assigns to each ordered triple (x,y,z) in a domain $D\subset\mathbb{R}^3$ (This means D is a subset of $\mathbb{R}^3$) a unique real number denoted by $f(x,y,z)$.

For example, the temperature on the surface of the earth depends on the longitude x, latitude y, and the time t.

$$\Rightarrow \text{Temp} = f(x,y,t)$$

This function may be even a function of more than three variables

$$\text{Temp} = f(x,y,z,t)$$

Example: Find the domain of f if $f(x,y,z)= \ln(z-y)+ xy\sin z$

$$z-y>0 \Rightarrow z>y$$ (in $\sin z$, z may have any value)

$$\Rightarrow D=\{(x,y,z)\in\mathbb{R}^3 \mid z>y\}$$ Read this " All (x,y,z) triples in $\mathbb{R}^3$ such that z is greater than y.

* Note: It is very difficult to visualize a function of three variables by its graph, since that would lie in a four-dimensional space. However we may get some insight into f by examining its level surfaces (Not level curves!) which are the surfaces w/ equations $f(x,y,z)=k$, where k is a constant.