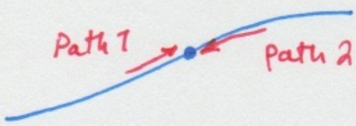
let's start from something that is familiar for us. previously, we have dealt with the limit of a single variable function. Now, let's think about the meaning of $\lim_{x \to a} f(x) = l$.

This means, if we approach to <u>a</u> along x-axis (this is the path that x can travel along), and follow the curve on the right side of <u>a</u>, do we get to the same height (function value, $f(x)$) as if we follow the curve on the left side of the point w/ $x=a$?

In other words, will we get L as the height, if we approach <u>a</u> from:

$\begin{cases} \text{Path 1: Left-hand Side} \\ \text{Path 2: Right-hand Side} \end{cases}$



Limit of multi-variable function:

Now, the question is, what should we have in order for the limit to exist for

$$\lim_{(x,y) \to (a,b)} f(x,y)$$

when we have a function of <u>one</u> variable, we have <u>one</u> path (along the curve)

when we have a function of <u>two</u> variables, we have <u>infinite</u> # of paths because we are actually traveling over a surface (The path that we choose is in the xy-plane but the curve that we travel along is in 3D space. The same way that when we are calculating the limit of single variable function, we travel along a curve $y=f(x)$ but it is only the x value that is changing along x axis.)

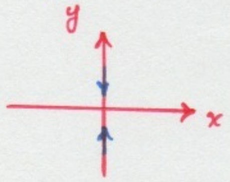
* Note: If you want to prove that the limit exists for multi-variable functions say $\lim_{(x,y) \to (a,b)} f(x,y)$ we need to prove that it exists along ALL the paths.

Very important Note: In general, for multi-variable functions, it is hard to prove that the limit exist, but it is easy (fairly) to prove that the limit does Not exist. All we need to do is, to pick two different paths and show that the height or the value of the function is different if we transverse along these different paths.

Example: Find $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{2x^2+y^2}$ or show that it does not exist (DNE)

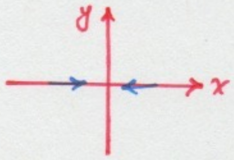
We choose two different paths to approach point $(0,0)$

1- move along y-axis (Force $x=0$)



$$\lim_{(x,y)\to(0,0)} f(0,y) = \lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} = -1$$

2- move along x-axis (Force $y=0$)



$$\lim_{(x,y)\to(0,0)} f(x,0) = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$$

As it can be seen in this problem by transversing along different paths when we are approaching $(0,0)$, we get two different values for $f(x,y)$. This means that the limit does not exist at $(0,0)$.

Just for practice, let's pick another path ($y=x$) and see what happens:

$y=x$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{2x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2-x^2}{2x^2+x^2} = \boxed{0}$$

$y=\sqrt{x}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{2x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2-x}{2x^2+x} = \boxed{-1}$$

(Note that for this one x can not approach to 0 from left-hand side $x \to 0^-$)

* Make sure the path you choose contains the point (a,b).
  e.g. $(0,0)$ is on $y=x$, $y=x^2$, $y=2x$, ...
* It is highly recommended that one of the paths you choose is either by forcing $x=0$ and moving along y-axis or forcing $y=0$ and moving along x-axis.
* It is helpful to choose a path that makes the degree of the numerator and the denominator equal.

Example: find the $\lim\limits_{(x,y)\to(0,0)} \frac{3xy}{3x^2+y^2}$

Path 1: Force $x=0$ and move along y-axis.

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{3x^2+y^2} = 0$$

Path 2: Force $y=0$, move along x-axis

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{3x^2+y^2} = 0$$

Does Not Help!

Path 3: Along $y=x$ $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2}{3x^2+x^2} = \frac{3}{4}$

The limit does Not exist!

Ex. Find the $\lim\limits_{(x,y)\to(0,0)} \frac{xy^3\cos x}{2x^2+y^6}$ or show that the limit DNE.

: Force $x=0$ and move along y-axis $\lim\limits_{(x,y)\to(0,0)} \frac{xy^3\cos x}{2x^2+y^6} = \lim\limits_{(x,y)\to(0,0)} \frac{0\,y^3\cos 0}{2(0)^2+y^6} = 0$

: Path $x=y^3$ (Take the one that has smaller power and make it to the larger power)

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{y^6\cos y^3}{2y^6+y^6} = 1/3$$

Therefore, the limit does not exist.

Example: find the $\lim\limits_{(x,y)\to(1,0)} \dfrac{2xy-2y}{x^2+y^2-2x+1}$ or show that the limit DNE.

Path 1: Force $x=1$ and move along y-axis

$$\lim_{(x,y)\to(0,0)} \frac{2y-2y}{1+y^2-2+1} = 0$$

Path 2: move along $x=y+1$ or $y=x-1$ (Notice that $(1,0)$ is on the path)

$$\lim_{(x,y)\to(0,0)} \frac{2xy-2y}{x^2+y^2-2x+1} = \lim_{(x,y)\to(0,0)} \frac{2(y+1)y-2y}{(y+1)^2+y^2-2(y+1)+1} = \lim_{(x,y)\to(0,0)} \frac{2y^2+2y-2y}{2y^2} = 1$$

Therefore, the limit DNE

* Note: There are ways to calculate the limit of multi-variable functions (instead of trying to prove that the limit does not exist)
If you see a lot of terms like $x^3, y^3, x^2, y^2, \dots$ you might be able to convert x, y in Cartesian coordinate system to Polar coordinates.
(use $x^2+y^2=r^2$, $x=r\cos\theta$, $y=r\sin\theta$)

Example: Find the $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2}$ or prove that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to0^+} \frac{r^3\cos^3\theta+r^3\sin^3\theta}{r^2\cos^2\theta+r^2\sin^2\theta} = \lim_{r\to0^+} \frac{r^3(\sin^3\theta+\cos^3\theta)}{r^2}$$

$$= \lim_{r\to0^+} r(\sin^3\theta+\cos^3\theta) = 0$$

So the limit does exist and is equal to $\emptyset$.

Example: Find the $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$ or prove the limit DNE.

$x^2+y^2=r^2$

$$\Rightarrow \lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0^+} r^2 \ln r^2 = 0\times(-\infty)$$

Indeterminate form $0\times\infty$

$$= \lim_{r\to 0^+} \frac{\ln r^2}{1/r^2} = \frac{-\infty}{\infty} \xrightarrow{\text{L'Hospital's}} = \lim_{r\to 0^+} \frac{\frac{2r}{r^2}}{\frac{0-2r}{r^4}} = \lim_{r\to 0^+} \frac{2r^5}{-2r^3} = 0$$

Continuity:

Sometimes we can use the concept of continuity for proving that the limit exists (And hence, obtain the value of the limit)

* If we have a region (two dimensional region in xy-plane) that the function is defined in that region, the function is continuous in that region. In other words, a function is continuous at any point inside its domain.
* Polynomials are continuous everywhere.
* Rational functions are continuous everywhere except where the denominator is zero.

Example: $f(x,y)=\dfrac{x^3+xy+y^3}{x^2+y^2+1}$ is continuous everywhere. $f(x,y)=\dfrac{x^3+xy+y^3}{x^2+y^2}$ is continuous everywhere except at $(0,0)$

Example: Find $\lim\limits_{(x,y)\to(0,1)} \dfrac{x^3+xy+y^3}{x^2+y^2+1}$ or prove that the limit DNE.

Since $f(x,y)=\dfrac{x^3+xy+y^3}{x^2+y^2+1}$ is continuous everywher (in $\mathbb{R}^2$) therefore, the limit exists and is equal to the value of the function at the point.

$$\Rightarrow f(0,1) = \frac{0+0+1}{0+1+1} = 1/2 \Rightarrow \lim_{(x,y)\to(0,1)} f(x,y) = 1/2$$

Example: Investigate the continuity of $f(x,y)=\sqrt{x}\, e^{x/y}$

$x \geqslant 0$ (under the square root we can not have a negative number)

$y \neq 0$ (because then $e^{x/y}$ blows up and goes to infinity)

Squeeze Theorem (This is not part of your syllabus and you wont be asked about this in your exams)

Example: Find the limit of the following function or prove that the limit DNE.

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$$

$$0 \leqslant \frac{x^2}{x^2+y^2} \leqslant 1$$

we know that $0 \leqslant \frac{5x^2}{x^2+y^2} \leqslant 5 \Rightarrow 0 \leqslant \frac{5x^2|y|}{x^2+y^2} \leqslant 5|y|$

* Note: we can not multiply the inequality by y because if $y<0$, then the direction of inequality (signs) changes. So we multiply it by $|y|$

$$0 \leqslant \frac{5x^2|y|}{x^2+y^2} \leqslant 5|y| \Rightarrow 0 \leqslant \left|\frac{5x^2y}{x^2+y^2}\right| \leqslant 5|y|$$

$$\lim_{(x,y)\to(0,0)} 0 = 0 \quad , \quad \lim_{(x,y)\to(0,0)} 5|y| = 0 \xRightarrow[\text{Theorem}]{\text{By Squeeze}} \lim_{(x,y)\to(0,0)} \left|\frac{5x^2y}{x^2+y^2}\right| = 0$$

So far we have proved $\lim\limits_{(x,y)\to(0,0)} \left|\frac{5x^2y}{x^2+y^2}\right| = 0$ but we are after $\lim\limits_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$

From Math 2A you remember "The limit of composite functions theorem": If g is continuous at $\underline{a}$ and f is continuous at $g(a)$ $\Rightarrow \lim\limits_{x\to a} f(g(x)) = f(\lim\limits_{x\to a} g(x))$

We know that the absolute value function is continuous. Therefore:

$$\lim_{(x,y)\to(0,0)} \left|\frac{5x^2y}{x^2+y^2}\right| = \left|\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}\right| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0$$