section 14.3

In Math 2A we learned that the derivative of a single variable function like $f(x)$ is representative of instantaneous rate of change of function f along x-axis. We also learned that the derivative of a function at point $x=a$ can be calculated using:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \text{and} \quad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

By replacing $h = x-a$ we get

We also learned that the derivative of a function can be also represented as another function, $f'(x)$, and we defined it as:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative of functions of two variables (and more) can be defined in a similar way.

* If a function of two variables x, and y like $f(x,y)$ is given, the derivative of f can be calculated by fixing y and setting it equal to a constant $(y=b)$, and varying x. If this is done, we are really taking the derivative of a single variable function, $g(x)$, such that $g(x) = f(x,b)$.

Now, if g has a derivative at $\underline{a}$, then we call it a partial derivative of f w/ respect to x, at (a,b) and we denote it w/ $f_x(a,b)$:

$$\boxed{\begin{array}{l} f_x(a,b) = \lim\limits_{h\to 0} \dfrac{f(a+h,b) - f(a,b)}{h} \\ \hline f_y(a,b) = \lim\limits_{h\to 0} \dfrac{f(a,b+h) - f(a,b)}{h} \end{array}}$$

we can also define the partial derivatives of $f(x,y)$ with respect to independent variables

$$\boxed{\begin{array}{l} f_x(x,y) = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h} \\ \hline f_y(x,y) = \lim\limits_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h} \end{array}}$$

we use different notation for partial derivative of a function with two variables

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

because we mostly replace $f(x,y)$ with z

we read $\frac{\partial f}{\partial x}$, "df over dx" but we write it as d is melting down! This is the way that we distinguish b/w the derivative of a function and the partial derivative of a multi-variable function. Similarly:

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Very Important Note: In reality, when we have a function w/ Two variables like $f(x,y)$, in order to calculate partial derivatives, we Just apply the rules of differentiation.

1- To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.

2- To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Example: If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$

$$f_x(x,y) = 3x^2 + 2xy^3 \Rightarrow f_x(2,1) = 2(2)^2 + 2(2)(1)^3 = 16$$

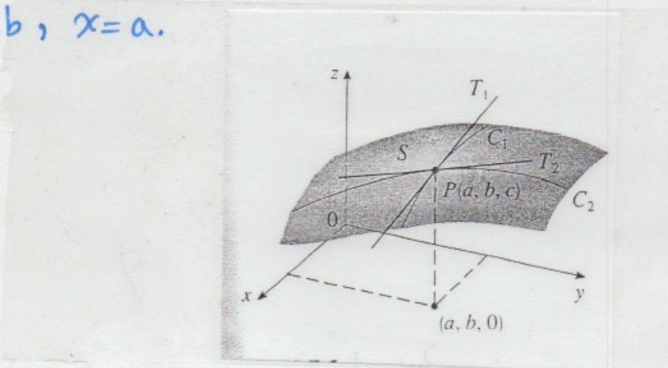
$$f_y(x,y) = 3x^2y^2 - 4y \Rightarrow f_y(2,1) = 3(2)^2(1)^2 - 4(1) = 8$$

Geometric Interpretation of partial derivatives:

we recall that $z = f(x,y)$ represents a surface S. If we pick a point $P(a,b)$ on surface S, the intersection of surface S and vertical plane $y=b$ gives C_1 curve. likewise, by fixing x and setting $x=a$, the vertical plane $x=a$ intersects surface S at a curve C_2 (Note that both these curves pass through the point $P(a,b)$). Also, Note that C_1 is the graph of $g(x) = f(x,b)$ and curve C_2 is the graph of $h(y) = f(a,y)$.

This means the slope of the tangent line T_1 at point $P(a,b)$ is $g'(a)$ and the slope of tangent line T_2 at point $P(a,b)$ is $h'(b)$.

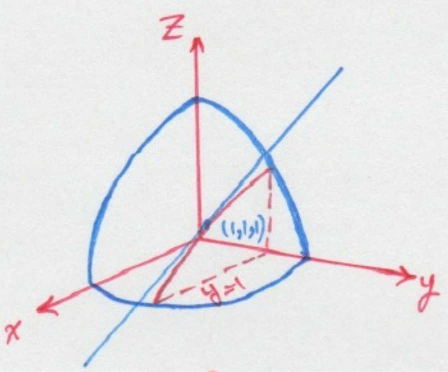
Therefore, the partial derivative of $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as the slope of tangent lines at point P to the traces C_1 and C_2 in the planes $y=b$, $x=a$.



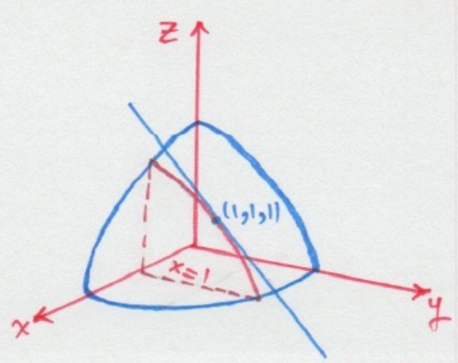
Example: If $f(x,y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers.

$$\begin{cases} f_x(x,y) = -2x \Rightarrow f_x(1,1) = -2 \\ f_y(x,y) = -4y \Rightarrow f_y(1,1) = -4 \end{cases}$$

So the slope of the tangent line along x axis is -2 and along y axis is -4



slope $f_x(1,1) = -2$



slope: $f_y(1,1) = -4$

Example: If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right)\frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right)\frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = \frac{-x}{(1+y)^2}\cdot\cos\left(\frac{x}{1+y}\right)$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation: $x^3+y^3+z^3+6xyz=1$

Notice that z is a function of x and y. $z=f(x,y)$

$\frac{\partial z}{\partial x} \rightarrow$ we take the partial derivative from both sides w/ respect to x

$$\Rightarrow 3x^2+0+\frac{\partial}{\partial x}(z^3)+6y\cdot\frac{\partial}{\partial x}(xz) = \frac{\partial}{\partial x}(1)$$

$$\Rightarrow 3x^2+\underbrace{\frac{\partial}{\partial z}(z^3)\cdot\frac{\partial z}{\partial x}}_{\text{The chain Rule}}+6y\underbrace{\left(z\cdot\frac{\partial x}{\partial x}+x\frac{\partial z}{\partial x}\right)}_{\text{The product rule}}=0$$

$$\Rightarrow 3x^2+3z^2\cdot\frac{\partial z}{\partial x}+6y\left(z+x\frac{\partial z}{\partial x}\right)=0$$

$$\Rightarrow \frac{\partial z}{\partial x}(3z^2+6xy) = -3x^2-6yz \Rightarrow \frac{\partial z}{\partial x} = \frac{-3x^2-6yz}{3z^2+6xy} = \frac{-(x^2+2yz)}{z^2+2xy}$$

Similarly:

$$\frac{\partial z}{\partial y} \longrightarrow 3y^2+3z^2\cdot\frac{\partial z}{\partial y}+6x\left(z+y\frac{\partial z}{\partial y}\right)=0$$

$$\Rightarrow \frac{\partial z}{\partial y}(3z^2+6xy) = -3y^2-6xz$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-(3y^2+6xz)}{3z^2+6xy} = \frac{-(y^2+2xz)}{z^2+2xy}$$

Functions of More than Two variables:

Partial derivatives can also be defined for functions of three or more variables, similar to functions of two variables.

e.g. if $w = f(x, y, z)$

* $f_x(x,y,z) = \lim\limits_{h \to 0} \dfrac{f(x+h, y, z) - f(x,y,z)}{h}$

* $f_y(x,y,z) = \lim\limits_{h \to 0} \dfrac{f(x, y+h, z) - f(x,y,z)}{h}$

* $f_z(x,y,z) = \lim\limits_{h \to 0} \dfrac{f(x, y, z+h) - f(x,y,z)}{h}$

We can not interpret these derivatives geometrically because the graph of f lies in four dimensional space.

Example: Find f_x, f_y, f_z if $f(x,y,z) = e^{xy} \ln z$.

$f_x = y e^{xy} \ln z$, $f_y = x e^{xy} \ln z$, $f_z = \dfrac{e^{xy}}{z}$

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x, and f_y are generally functions of two variable as well. Therefore, we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, $(f_y)_y$:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

In other words, the notation f_{xy} means, first we take the derivative w/ respect to x, and then we take the derivative w/ respect to y.

Example: Find the second partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$.

$f_x = 3x^2 + 2xy^3 \rightarrow f_{xx} = 6x + 2y^3$

$f_x = 3x^2 + 2xy^3 \rightarrow f_{xy} = 6xy^2$

$f_y = 3x^2y^2 - 4y \rightarrow f_{yy} = 6x^2y - 4$

$f_y = 3x^2y^2 - 4y \rightarrow f_{yx} = 6xy^2$

It seems that $f_{xy} = f_{yx}$. This is NOT a coincident.

Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a,b). If the function f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Example: Calculate f_{xxyz} if $f(x,y,z) = \sin(3x + yz)$.

$$f_x = 3\cos(3x+yz)$$

$$f_{xx} = -9\sin(3x+yz)$$

$$f_{xxy} = -9z\cos(3x+yz)$$

$$f_{xxyz} = -9\cos(3x+yz) - (-9\sin(3x+yz))yz$$

$$= 9yz\sin(3x+yz) - 9\cos(3x+yz)$$

* Notice that in $f_{xxy} = -9z\cos(3x+yz)$, the function is in the form of $g(z) \cdot h(z)$, so we have to use the product rule.

Partial Differential Equations:

A partial differential equation is a differential equation that is involved w/ partial derivatives of a function. A differential equation is an equation That shows the relationship between the function of one variable and its derivatives (might be involved with the independent variable as well). (e.g. $y' + y = \emptyset$)
When we solve a differential equation (whether The ordinary differential eq. or partial differential eq.) The solution is <u>not</u> a variable like x or t but it is a function such as y, $f(x)$, $f(x,y)$, ...

Partial differential equations play a crucial role in many branches of physics, engineering and mathematics.

Examples: Vibration, wave analysis, heat conduction, fluid dynamics, Aerodynamics, and heat transfer.

Two of The very important partial differential equations that you will be dealing w/ in your future courses (during your undergraduate / graduate years) are:

Laplace Equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Wave Equation: $$\underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{Time}} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Example: show that the function $u(x,y) = e^x \sin y$ is a solution of Laplace's Equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad : \quad \frac{\partial u}{\partial x} = e^x \sin y \,, \quad \frac{\partial^2 u}{\partial x^2} = e^x \sin y \quad ①$$

$$\frac{\partial u}{\partial y} = e^x \cos y \,, \quad \frac{\partial^2 u}{\partial y^2} = - e^x \sin y \quad ②$$

$$① + ② \longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y - e^x \sin y = \emptyset$$

Example: Verify that the function $u(x,y) = \sin(x - at)$ satisfies the wave equation.

$$\frac{\partial u}{\partial t} = -a \cos(x - at) \Rightarrow \frac{\partial^2 u}{\partial t^2} = -a^2 . \sin(x - at)$$

$$\frac{\partial u}{\partial x} = \cos(x - at) \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\sin(x - at)$$

$$\frac{\partial^2 u}{\partial t^2} = -a^2 \sin(x - at) = a^2\left(-\sin(x - at)\right) = a^2 . \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial^2 x}$$ which means $u(x,y) = \sin(x - at)$ Satisfies the eq. of wave.