One of the most important ideas of single variable calculus is that as we zoom in towards a point on the graph of a differentiable function, The graph becomes indistinguishable from its tangent line and we can approximate the function by a linear function.

In this section a similar idea can be developed in three dimesions. As we zoom in towards a point on a surface of a graph of a differentiable function of two variables, the surface looks more and more like a plane (The tangent plane) and we can approximate The function by a linear function of two variables.

Tangent plane:

In The previous section we saw that if The vertical planes $x = x_0$ and $y = y_0$ intersect w/ surface S at point (x_0, y_0, z_0), two curves are created, C_1 and C_2. We can find The tangent lines to the curves and call Them T_1, and T_2.

The Tangent Plane to The surface S at $P(x_0, y_0, z_0)$ is The plane that contains both tangents T_1, and T_2. Now, we want to find the eq. of This tangent plane!

We know That the general eq. for a plane is:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\Rightarrow z - z_0 = \left(-\frac{a}{c}\right)(x-x_0) + \left(-b/c\right)(y-y_0)$$

for simplicity, let's call $-a/c = m_1$ and $-b/c = m_2$

$$\Rightarrow z - z_0 = m_1(x-x_0) + m_2(y-y_0)$$

@ $y = y_0$, the eq. of The plane becomes $z - z_0 = m_1(x-x_0)$

we also know That the eq. of T_1 is also $z - z_0 = f_x(x_0, y_0)(x-x_0)$

$\Rightarrow m_1 = f_x(x_0, y_0)$

Similarly, The intersection of $x = x_0$ with the surface lead to creation of C_2, and its tangent T_2 with the eq.: $z - z_0 = f_y(x_0, y_0)(y - y_0)$

Also in the plane eq. if we set $x = x_0$ we get $z - z_0 = m_2(y - y_0)$ and we can see that $m_2 = f_y(x_0, y_0)$

The eq. of the tangent plane becomes

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Example: Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.

Let $f(x,y) = 2x^2 + y^2$

$f_x(x,y) = 4x \longrightarrow f_x(1,1) = 4$

$f_y(x,y) = 2y \longrightarrow f_y(1,1) = 2$

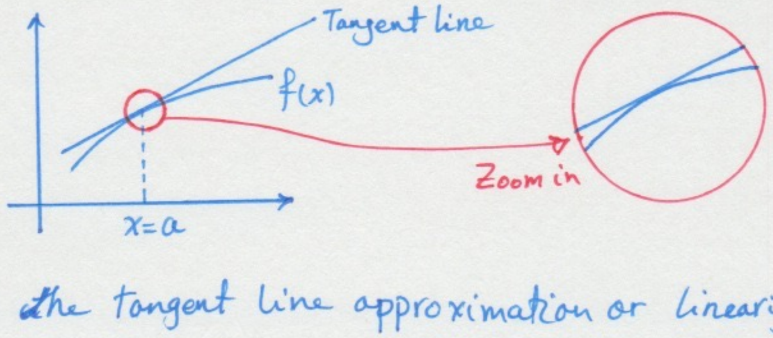
$\Rightarrow$ The eq. of the tangent plane is

$z - 3 = 4(x-1) + 2(y-1)$

$\Rightarrow z - 4x - 2y + 3 = 0$

Linear Approximation:

From Math 2A, you remember that we can approximate a single variable function in the vicinity of point $x = a$ by tangent line to the curve at point a.



This was called the tangent line approximation or Linearization of $f(x)$ around a.

we have a similar concept for the surfaces in the three dimensional coordinate system.

In the vicinity of point $P(a, b, f(a,b))$, the surface can be closely approximated by the tangent plane at point (a,b).

$z - z_0 = f_x\,(x-x_0) + f_y\,(y-y_0)$ at point $(a, b, f(a,b))$:

$$z - z_0 = f_x \cdot (x - x_0) + f_y \cdot (y - y_0)$$

we know $\left.\begin{array}{l} f(a,b) = z_0 \\ a = x_0 \\ b = y_0 \end{array}\right\} \Rightarrow \boxed{z - f(a,b) = f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b)}$

$$f(x,y) = z \approx L(x,y) = f(a,b) + f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b)$$

Compare this with the familiar formula that we had in math 2A

$$L(x) = f(a) + f'(a)(x-a)$$

* Note: The linear approximation/Tangent plane approximation only works for differentiable functions. The following Theorem discusses the conditions for a differentiable function.

Theorem: If partial derivatives of f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Example: show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ and find its linearization There. Then Approximate $f(1.1, -0.1)$.

$f_x(x,y) = e^{xy} + xye^{xy} \longrightarrow f_x(1,0) = e^0 + 0 = 1$

$f_y(x,y) = xe^{xy} \longrightarrow f_y(1,0) = 1 \times e^0 = 1$

Therefore, f_x and f_y both exist near $(1,0)$ and both are continuous, therefore, $f(x,y)$ is differentiable.

Tangent Plane $z - 1 = 1(x-1) + 1(y-0) \Rightarrow z = x + y$ or $\boxed{L(x,y) = x + y}$

This means in the vicinity of $(1,0)$, $xe^{xy} = x + y$

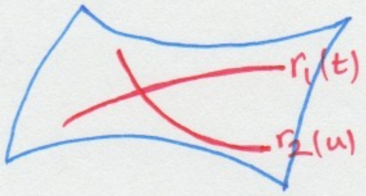
$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$, the actual value of $f(1.1,-0.1)$ is $1.1e^{(-0.1)(1.1)} \approx 0.985$

* Note: The concept of linear approximations can be defined in a similar manner for functions of more than two variables.

A differentiable function is defined by an expression similar to the one that we came up w/ for a function of two Variables. For such function, the Linear approximation is:

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

Important Example: Suppose that you need to know an eq. of the tangent plane to a surface at the point $(2,1,3)$. You do not have the eq. for S but you know that $r_1(t) = \langle 2+3t, 1-t^2, 3-4t+t^2 \rangle$ and $r_2(u) = \langle 1+u^2, 2u^3-1, 2u+1 \rangle$ both lie on S. Find the eq. of the tangent plane at P.



$r_1(t) = \langle 2+3t, 1-t^2, 3-4t+t^2 \rangle$

$x: 2+3t=2 \Rightarrow t=0$

Now, let's make sure that y, and z values are the same as $(2,1,3)$ for $t=0$

$t=0 \Rightarrow 1-t^2 = 1-0 = 1$ ✓

$t=0 \Rightarrow 3-4t+t^2 = 3$ ✓

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$r_2(u) = \langle 1+u^2, 2u^3-1, 2u+1 \rangle$

$z: 2u+1 = 3 \Rightarrow u=1$

$u=1 \Rightarrow 1+u^2 = 2$ ✓

$u=1 \Rightarrow 2u^3-1 = 1$ ✓

Now, we can say that r_1 and r_2 intersect at $(2,1,3)$. The point of intersection is $(2,1,3)$.

Now, if we find the direction of Tangent lines T_1, and T_2 on curves C_1, and C_2, the cross-product of those give us the Normal vector of the tangent plane

$r_1'(t) = \langle 3, -2t, 2t-4 \rangle \Rightarrow r_1'(0) = \langle 3, 0, -4 \rangle$

$r_2'(u) = \langle 2u, 6u^2, 2 \rangle \Rightarrow r_2'(1) = \langle 2, 6, 2 \rangle = 2\langle 1,3,1 \rangle$

$$r_1'(0) \times r_2'(1) = \begin{vmatrix} i & j & k \\ 3 & 0 & -4 \\ 1 & 3 & 1 \end{vmatrix} = 24i - 14j + 18k$$ (The normal vector of the tangent plane)

$\Rightarrow$ The plane eq. $24(x-2) - 14(y-1) + 18(z-3) = 0$

$\Rightarrow 12x - 7y + 4z = 44$