The Chain Rule

section 14.5

In this section, we want to take a closer look to partial derivatives and the chain rule. Let's start from an example from single variable calculus.

Suppose that we have a composite function like $f(g(x))$ and we know that This means, f is a function of g and g is a function of x.

The question is, "what is the meaning of $\frac{df}{dx}$" ?

we have the following diagram which again, means that f is a function of g and g is a function of x.

The rate of change of f with respect to x is the rate of change of f with respect to g multiplied by the rate of change of g with respect to x.

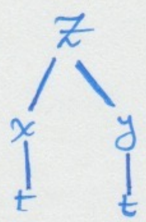
$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

In other words, we can say x changes g and g changes f.

The process of taking derivative of a multi variable function is very similar. by sketching the tree diagram, we can easily write down the chain rule for a function of Two variables or more:

Case 1: Theorem: Suppose that $z = f(x,y)$ is a differentiable function of x, and y. where $x = g(t)$, and $y = g(t)$ are both differentiable functions of t.

Then: $$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$



we use d here instead of ∂ because x, and y are both single variable functions of t.

Example: If $z = x^2y + 3xy^4$ where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = \emptyset$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = (2xy + 3y^4)\frac{dx}{dt} + (x^2 + 12xy^3)\frac{dy}{dt}$$

$$\frac{dz}{dt} = (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

$$t = \emptyset \Rightarrow \begin{cases} x = 0 \\ y = 1 \end{cases} \Rightarrow \left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2) + (0+0)(0) = \boxed{6}$$

Example: The pressure P (in kilopascals), Volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the Temperature is $300°K$, and increasing at a rate of $0.1\ K/s$ and the Volume is $100\ l$ and increasing at a rate of $0.2\ l/s$.

$$T = 300, \quad \frac{dT}{dt} = 0.1, \quad V = 100, \quad \frac{dV}{dt} = 0.2$$

$$P = 8.31\frac{T}{V} \Rightarrow \frac{\partial P}{\partial t} = \frac{\partial P}{\partial T} \cdot \frac{dT}{dt} + \frac{\partial P}{\partial V} \cdot \frac{dV}{dt}$$

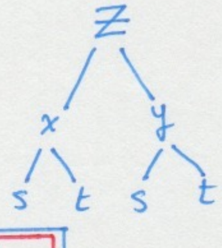
$$\Rightarrow \frac{\partial P}{\partial t} = \frac{8.31}{V} \cdot (0.1) + 8.31 \cdot T\left(-\frac{1}{V^2}\right)(0.2)$$

$$\frac{\partial P}{\partial t} = \frac{8.31 \times 0.1}{100} + \frac{8.31 \times 300 \cdot (0.2)}{-(100)^2} \approx -0.04155$$

The pressure is decreasing at the rate of $0.042\ kPa/s$

Case 2: Theorem: Now, let's try to find the derivative of a multivariable function $z = f(x,y)$ such that $x = g(s,t)$ and $y = h(s,t)$. In other words f is a function of x and y, while x, and y are functions of s, t.

Think about it this way: s causes a change in both x and y. x, and y cause a change in z and t.



$$\boxed{\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \end{aligned}}$$

Example: If $z = e^x \sin y$, where $x = st^2$, and $y = s^2 t$, find $\frac{\partial z}{\partial s}$, and $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial x} = e^x \sin y$$

$$\Rightarrow \frac{\partial z}{\partial s} = e^x \sin y\,(t^2) + e^x \cos y\,(2st) \qquad \frac{\partial z}{\partial y} = e^x \cos y$$

$$\Rightarrow \frac{\partial z}{\partial s} = t^2 e^{st^2}\cdot \sin(s^2 t) + 2st\, e^{st^2} \cos(s^2 t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = e^x \sin y\,(2st) + e^x \cos y\,(s^2)$$

$$\Rightarrow \frac{\partial z}{\partial t} = 2st\cdot e^{st^2}\cdot \sin(s^2 t) + s^2\cdot e^{st^2} \cos(s^2 t)$$

Note: In these examples, s, and t are considered as the independent variables, x, and y are considered the intermediate Variables, and z is considered the dependent variable.

The concept of the chain rule can be generalized as the following:

The Chain Rule (General Version): Suppose that u is a differentiable function of the n variables $x_1, x_2, \dots x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \dots, t_m$. Then

$u = f(x_1, x_2, \dots, x_n)$

$x_j = g(t_1, t_2, \dots, t_m)$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_i} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_i}}$$

Example: Write out the chain rule for the case where $w = f(x, y, z, t)$, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $t = t(u, v)$.

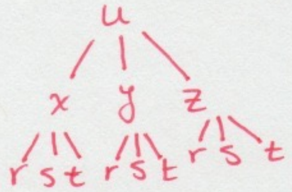
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial v}$$

Example: If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \cdot \sin t$

find $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = \emptyset$.

Let's sketch the tree diagram in this case



$$\Rightarrow \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial s} = 4x^3 y \,(re^t) + (x^4 + 2yz^3)(2rse^t) + (3z^2 y^2)(r^2 \sin t)$$

$$\left.\begin{aligned} x(r,s,t) &= x(2,1,\emptyset) = 2 \\ y(r,s,t) &= y(2,1,\emptyset) = 2 \\ z(r,s,t) &= z(2,1,\emptyset) = \emptyset \end{aligned}\right\} \quad \frac{\partial u}{\partial x} = 4(2)^3 \cdot (2)(2e^0) + (2^4 + 2(2)(0))(2 \times 2 \times 1 \times 1) + 0$$

$$= 192$$

Example: If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the eq. $t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = \emptyset$

Let $x = s^2 - t^2$, $y = t^2 - s^2$

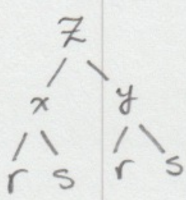
$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \Rightarrow \frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\Rightarrow \frac{\partial g}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} \Rightarrow \frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

$$\Rightarrow t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = t\left(\frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)\right) + s\left(\frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(-2s)\right)$$

$$= 0$$

Example: If $z = f(x,y)$ has continuous second order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find $\frac{\partial z}{\partial r}$, and $\frac{\partial^2 z}{\partial r^2}$.



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot (2r) + \frac{\partial z}{\partial y} \cdot (2s) \Rightarrow \quad \frac{\partial z}{\partial r} = 2r \frac{\partial z}{\partial x} + 2s \frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(2r \cdot \frac{\partial z}{\partial x} + 2s \frac{\partial z}{\partial y}\right)$$

$$= 2 \frac{\partial z}{\partial x} + 2r \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 0 + 2s \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) \quad (1)$$

We go ahead and apply the chain rule again:

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial y}{\partial r} = 2r \cdot \frac{\partial^2 z}{\partial x^2} + 2s \frac{\partial^2 z}{\partial y \partial x} \quad (2)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \cdot \frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x \partial y} \cdot (2r) + \frac{\partial^2 z}{\partial y^2} \cdot 2s \quad (3)$$

We put expressions (1), (2), and (3) Together and we will get

$$\frac{\partial^2 z}{\partial r^2} = 2 \frac{\partial z}{\partial x} + 4r^2 \frac{\partial^2 z}{\partial x^2} + 8rs \frac{\partial^2 z}{\partial x \partial y} + 4s^2 \frac{\partial^2 z}{\partial y^2}$$

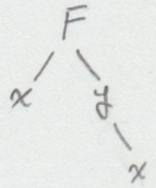
Implicit Differentiation:

Previously, you have seen the process of implicit differentiation. In this section we want to take a closer look at the process. Suppose that an equation of form $F(x,y)=0$ defines y implicitly as a differentiable function of x, that is, $y=f(x)$ where $F(x,y)=0$ or $F(x,f(x))=0$

The chain rule for this equation can be written as: (suppose that x is changing and we want to see the rate of change of F wrt x)

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0 \Longrightarrow \frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{F_x}{F_y}$$



Example: Find y' if $x^3+y^3=6xy$

$$F(x,y)=x^3+y^3-6xy=0$$

$$y'=\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{3x^2-6y}{3y^2-6x}=\frac{2y-x^2}{y^2-2x}$$

* Note: The implicit differentiation formula may be applied to functions of several variables as well. Suppose z is given implicitly as $z=f(x,y)$ by an eq. of Form $F(x,y,z)=\emptyset$

we know that $\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}=\emptyset$ $\left\{\text{we know } \frac{\partial}{\partial x}(x)=1,\ \frac{\partial}{\partial x}(y)=\emptyset\right.$

$$\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}=0 \Longrightarrow \frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

and

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$

$$\Rightarrow F(x,y,z) = x^3+y^3+z^3+6xyz-1 = \emptyset$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$$