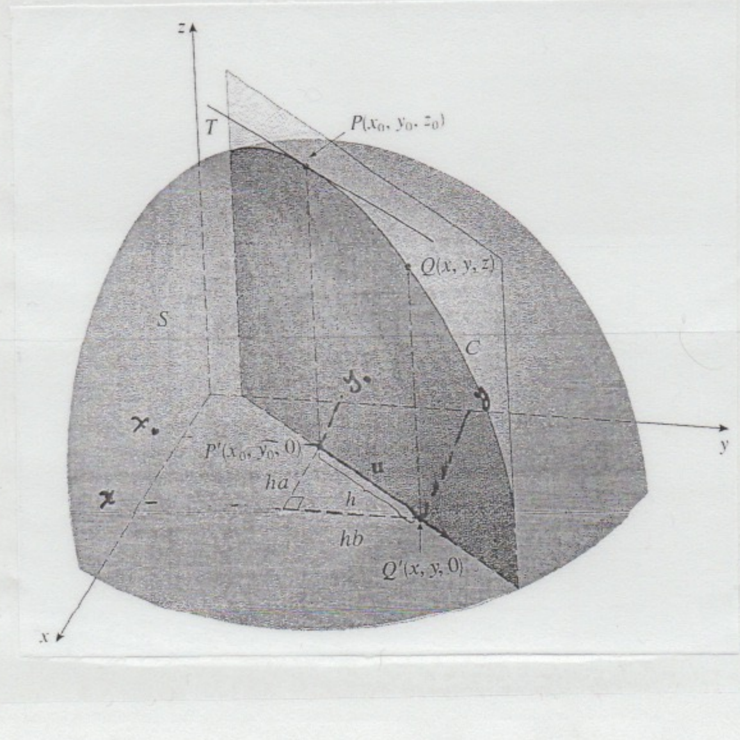
We have already discussed the concept of directional derivatives, The partial derivatives are directional derivatives, but very specific ones. The partial derivatives are directional derivatives along x_axis and y_axis. Now, we want to extend this idea. How do we find the directional derivatives along any given direction?

We need a direction to define the directional derivative, say $\vec{u} = \langle a, b \rangle$, where $\vec{u}$ is a unit vector. (we do not want the magnitude of the vector changes the slope of the derivative).

Note: the vector $\vec{u}$ needs to be in xy_plane (The same way that x-axis and y-axis are in xy_plane). Also note that we can find the directional derivative along a 3D vector in space, $u = \langle a, b, c \rangle$, but we do not discuss it here at this point.

Suppose that now we wish to find the rate of change of $z = f(x,y)$ at (x_0, y_0) in the direction of an arbitrary unit vector $\vec{u} = \langle a, b \rangle$. we know that $P(x_0, y_0, z_0)$ lies on the surface S. The vertical plane that passes through point P and is parallel to u, intersects S in a curve C. The slope of the tangent line at point $P(x_0, y_0, z_0)$ is the rate of change of z in the direction of $\vec{u}$.



If we pick another point on curve C, say $Q(x, y, z)$ and look at the projection of PQ curve onto xy-plane, we see $P'Q'$. From the figure we see that $\overrightarrow{P'Q'}$ is along the same direction as $\vec{u}$ but has a different magnitude:

$$\overrightarrow{P'Q'} = h\vec{u} = \langle ha, hb \rangle \quad (h \text{ is a scalar number})$$

If we find the magnitude of projection of $\overrightarrow{P'Q'}$ along x-axis and y-axis ($\text{comp}_i \overrightarrow{P'Q'}$, $\text{comp}_j \overrightarrow{P'Q'}$) we see:

$$\begin{cases} x - x_0 = ha \\ y - y_0 = hb \end{cases} \Rightarrow \begin{cases} x = x_0 + ha \\ y = y_0 + hb \end{cases}$$

$$\Delta z = z - z_0 = f(x, y) - f(x_0, y_0) \Rightarrow \frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we calculate the limit when $h \to 0$, we obtain the rate of change of z in the direction of $\vec{u}$.

Theorem: The directional derivative of f at (x_0, y_0) in the direction of unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \quad \text{if the limit exists.}$$

In general, we mostly use the following theorem for our calculations:

Theorem: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any vector $\vec{u} = \langle a, b \rangle$ and:

$$D_u f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

You can find the proof of this theorem in page 948 and 949 of your book.

Note: If the unit vector makes an angle θ w/ the positive x-axis, then we can write $\vec{u} = \langle \cos\theta, \sin\theta \rangle$ (Notice that u is a unit vector too)

Example: Find the directional derivative $D_u f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and u is the unit vector given by the angle $\theta = \frac{\pi}{6}$. What is $D_u f(1,2)$?

$f_x = 3x^2 - 3y$

$f_y = 8y - 3x$

$\vec{u} = \langle \cos\theta, \sin\theta \rangle = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$D_u f = f_x a + f_y b$

$D_u f = (3x^2 - 3y)\frac{\sqrt{3}}{2} + \frac{1}{2}(8y - 3x)$

$\Rightarrow D_u f(1,2) = (3\times1^2 - 3\times2)\frac{\sqrt{3}}{2} + (8\times2 - 3)\frac{1}{2} = \frac{13 - 3\sqrt{3}}{2}$

The Gradient Vector

Notice from The previous theorem $D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$ can be written in terms of dot product of two vectors:

$D_u f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle = \langle f_x, f_y \rangle \cdot \vec{u}$

The first vector in dot product, $\langle f_x, f_y \rangle$, occurs frequently in Mathematics. Therefore, we are going to give it a special name The Gradient Vector, and we will use a special notation for it, grad f, or ∇f, which is read "del f"

Definition:

If f is a function of two variables x, and y, then the gradient of f is a vector function defined by:

$$\nabla f = \nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

Example: If $f(x,y) = \sin x + e^{xy}$ find $\nabla f(0,1)$.

$\nabla f = \langle f_x, f_y \rangle = \langle \cos x + y e^{xy}, x e^{xy} \rangle$

$\nabla f(0,1) = \langle \cos 0 + 1 e^{0\times1}, 0 e^{0\times1} \rangle = \langle 2, 0 \rangle$

Note: Using this new notation, we can write $D_u f(x,y) = \nabla f(x,y) \cdot \vec{u}$

Example: Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of vector $v = 2i + 5j$.

$v = 2i + 5j$ is not a unit vector $\Rightarrow \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2i+5j}{\sqrt{4+25}} = \frac{2}{\sqrt{29}} i + \frac{5}{\sqrt{29}} j$

$\nabla f = \langle f_x, f_y \rangle = \langle 2xy^3, 3x^2y^2 - 4 \rangle$

$\nabla f(2,-1) = \langle 2(2)(-1)^3, 3(2)^2(-1)^2 - 4 \rangle = \langle -4, 8 \rangle$

$D_u f(2,-1) = \nabla f \cdot \vec{u} = \langle -4, 8 \rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle \Rightarrow D_u f(2,-1) = \frac{-8}{\sqrt{29}} + \frac{40}{\sqrt{29}}$

$$\Rightarrow \boxed{D_u f(2,-1) = \frac{32}{\sqrt{29}}}$$

Functions of Three Variables:

for functions of three variables we can define directional derivatives in a similar manner. It can be interpreted as the rate of change of the function in the direction of a unit vector $\vec{u}$.

Definition: The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = \langle a, b, c \rangle$ is:

$$D_u f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$ if this limit exists.

Note: we can write all the previous formulas for a function of three variables x, y, and z.

$$D_u f(x,y,z) = f_x(x,y,z)\, a + f_y(x,y,z)\, b + f_z(x,y,z)\, c$$

&

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \cdot \vec{k}$$

$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$$

Example: If $f(x,y,z) = x \sin yz$. (a) Find the gradient of f and (b) Find the directional derivative of f at $(1,3,0)$ in the direction of $\vec{v} = i + 2j - k$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle \sin yz,\ xz \cos yz,\ xy \cos yz \rangle$$

$$\Rightarrow \nabla f(1,3,0) = \langle \sin \emptyset, \emptyset, 3 \rangle = \langle 0, 0, 3 \rangle$$

$v = i + 2j - k$ so the unit vector of v is $\vec{u} = \frac{1}{\sqrt{1+4+1}} i + \frac{2}{\sqrt{1+4+1}} j - \frac{1}{\sqrt{1+4+1}} k$

$$\vec{u} = \frac{1}{\sqrt{6}} i + \frac{2}{\sqrt{6}} j - \frac{1}{\sqrt{6}} k$$

$$\Rightarrow D_u f = \nabla f \cdot \vec{u} = \langle 0,0,3 \rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle = -\frac{3}{\sqrt{6}} = -\sqrt{3/2}$$

Maximizing the directional derivative:

Suppose we have a function of two or three variables and we consider all possible directional derivatives of f at a given point. These represent the rate of change of f in all possible directions. Now the question is, in which direction does f change fastest & what is the Max rate of change.

Theorem: If f is a differentiable function of two or three variables, the Max value of directional derivative is $|\nabla f|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector.

$$D_u f = \nabla f \cdot \vec{u} = |\nabla f| \cdot \underbrace{|\vec{u}|}_{\text{unit vector}} \cos\theta = |\nabla f| \cdot \cos\theta$$

So, max values of $D_u f$ is $|\nabla f|$ and it occurs when $\theta = 0$ or $\cos\theta = 1$.

Example: If $f(x,y) = xe^{y}$, find the rate of change of f at point $(2,0)$ in the direction of $\overrightarrow{PQ}$ with $Q(1/2, 2)$. In what direction does f have the maximum rate of change? What is the Max rate of change?

$\overrightarrow{PQ} = \langle \frac{1}{2} - 2, 2 - 0 \rangle = \langle -\frac{3}{2}, 2 \rangle$ This is not a unit vector and we have to find the unit vector in the direction of $\overrightarrow{PQ}$

$$\Rightarrow \vec{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \langle -\frac{3}{2}, 2 \rangle \Big/ \sqrt{\frac{9}{4} + 4} = \langle -\frac{3}{2}, 2 \rangle \frac{1}{\sqrt{\frac{25}{4}}} = \frac{2}{5} \langle -\frac{3}{2}, 2 \rangle$$

$$D_{\vec{u}} f(x,y) = \langle f_x, f_y \rangle \cdot \vec{u} = \langle e^{y}, xe^{y} \rangle \cdot \vec{u}$$

$$D_{\vec{u}} f(2,0) = \langle e^{0}, 2e^{0} \rangle \cdot \langle -\frac{3}{2}, 2 \rangle \frac{2}{5} = \langle 1, 2 \rangle \cdot \langle -\frac{3}{2}, 2 \rangle \frac{2}{5} = \frac{2}{5}\left(-\frac{3}{2} + 4\right)$$

$$= \frac{2}{5} \cdot \frac{5}{2} = 1 \quad \Rightarrow \boxed{D_{\vec{u}} f(2,0) = 1}$$

According to the Theorem, f increases fastest in the direction of the gradient vector $\nabla f(2,0) = \langle 1, 2 \rangle$ and the Max rate is $|\langle 1,2 \rangle| = \sqrt{5}$ (when $\theta = 0$ and $D_u f = |\nabla f|$)

Example: Suppose that the temperature at a point (x,y,z) in space is given by the following relationship: $T(x,y,z) = \dfrac{80}{1 + x^2 + 2y^2 + 3z^2}$, where T is measured in degrees celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1,1,-2)$? What is the Max rate of increase?

$$\nabla T = \left\langle \frac{-80(2x)}{(1+x^2+2y^2+3z^2)^2}, \frac{-80(4y)}{(1+x^2+2y^2+3z^2)^2}, \frac{-80(6z)}{(1+x^2+2y^2+3z^2)^2} \right\rangle$$

$$\Rightarrow \nabla T = \frac{160}{(1+x^2+2y^2+3z^2)^2} \langle -x, -2y, -3z \rangle, \quad \nabla T(1,1,-2) = \frac{160}{(1+1^2+2(1)^2+3(-2)^2)^2} \langle -1, -2, 6 \rangle$$

$$\Rightarrow \nabla T(1,1,-2) = \frac{5}{8}(-i - 2j + 6k)$$

The temperature increases fastest in this direction

$$\text{The Max rate} = |\nabla T| = \frac{5}{8}\sqrt{1+4+36} = \frac{5\sqrt{41}}{8} \approx 4^{\circ}\, C/m$$

Tangent planes to surfaces:

Suppose S is a surface w/ eq. $F(x,y,z)=k$, and let $P(x_0, y_0, z_0)$ be a point on S. let C be any curve that lies on the surface S and passes through the point P. C is a space curve that can be described by a continuous vector function $\vec{r}(t)=\langle x(t), y(t), z(t)\rangle$. Since curve C lies on surface S, any point $(x(t), y(t), z(t))$ must satisfy the eq. of the surface:

$$F(x(t), y(t), z(t))=k$$

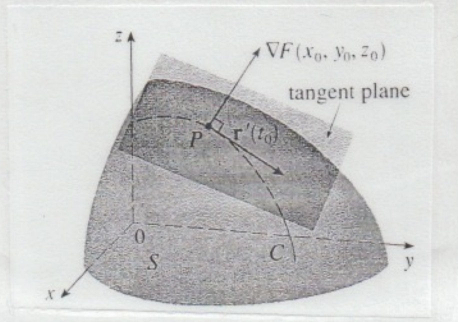
let's take the derivatives of both sides w/ respect to t (using the chain rule)

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial F}{\partial z}\cdot\frac{dz}{dt}=0 \Rightarrow \langle f_x, f_y, f_z\rangle\cdot\langle x'(t), y'(t), z'(t)\rangle=0$$

$$\Rightarrow \boxed{\nabla F\cdot r'(t)=0}$$

* Note: this equation says that at an arbitrary point on the surface, the gradient vector at that point is perpendicular to any tangent vector to any curve C on S that passes through P.

* Note: If $\nabla F\neq 0$ we can define the tangent plane to the surface $F(x,y,z)=k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has a normal vector $\nabla F(x_0, y_0, z_0)$. The eq. of this tangent plane would be in the form of:



$$\boxed{F_x(x_0, y_0, z_0)(x-x_0)+F_y(x_0, y_0, z_0)(y-y_0)+F_z(x_0, y_0, z_0)(z-z_0)=0}$$

The normal line to S at P is the line passing through P and perpendicular to the tangent plane. Again, the direction of the normal line is given by the gradient vector $\nabla F(x_0, y_0, z_0)$.

$$\boxed{\frac{x-x_0}{F_x(x_0, y_0, z_0)}=\frac{y-y_0}{F_y(x_0, y_0, z_0)}=\frac{z-z_0}{F_z(x_0, y_0, z_0)}}$$

Example: Find the equations of the tangent plane and normal line at the point $(-2,1,-3)$ to the ellipsoid:

$$\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$$

$$\nabla F(x,y,z)=\left\langle \frac{x}{2},\ 2y,\ \frac{2z}{9}\right\rangle \Rightarrow \nabla F(-2,1,-3)=\left\langle -1,\ 2,\ -\frac{2}{3}\right\rangle$$

Eq. of tangent plane: $-1(x+2)+2(y-1)-\frac{2}{3}(z+3)=0$

If we simplify this: $\boxed{3x-6y+2z+18=0}$

symmetric eqs. of the normal line are:

$$\frac{x+2}{-1}=\frac{y-1}{2}=\frac{z+3}{-2/3}$$