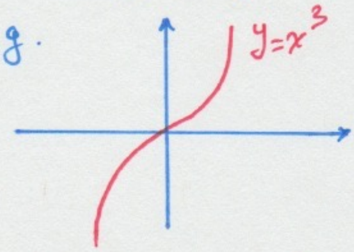
In Math 2A, we investigated the concept of Max and Min Values of a single variable function in details. In This section we go over the rules, definitions, and theorems of finding Max and Min value of a multivariable function of form $z = f(x,y)$. The process is very similar to what we learned in Math 2A but we just apply them on multivariable functions.

Let's go over the theorems and definitions for a single variable function $y = f(x)$

(This is a review from Math 2A)

Fermat's Theorem: If f has a local Max or local Min at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$.

In other words, if $f'(c) = 0$ or $f'(x) = 0$, we do not necessarily have a local Max or a local Min at $x = c$. e.g.

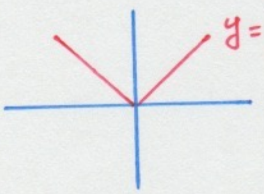


$f'(0) = 0$ but we do Not have a local Max or Min at $x = 0$

Definition: A critical number (critical point) of f is a number c in the domain of the function such that $f'(c) = 0$ or $f'(c)$ DNE. or $f'(x)$ DNE

Therefore, every Local Min or Max is a critical number and $f'(x) = 0$ at that point but of course, this is not correct in general. (Again you can have $f'(x) = 0$ for points that are not local Max or local Min)

Note: Example for a local Min for the case That $f'(x)$ does Not exist:



we have a local Min at $x = 0$ but $f'(0)$ does NOT exist.

The Second Derivative Test:

a) If $f'(c) = 0$ and $f''(c) > 0$, Then f has a local Min at c

b) If $f'(c) = 0$ and $f''(c) < 0$, Then f has a local Max at c

Now, let's apply similar concepts to use partial derivatives to locate Maxima and Minima of functions of two variables.

Definition: A function of Two variables has a local Maximum at (a,b) if $f(x,y) \leq f(a,b)$ when (x,y) is near (a,b). The number $f(a,b)$ is called a local Maximum Value. If $f(x,y) \geqslant f(a,b)$ when (x,y) is near (a,b), Then f has a local minimum at (a,b) and $f(a,b)$ is a local Minimum Value.

Theorem: If f has a local Max or Min at (a,b), and the first partial derivatives of f exists There, then $f_x(a,b) = \emptyset$ and $f_y(a,b) = \emptyset$.

(This Theorem is Similar to Fermat's Theorem)

Note: A point (a,b) is called a critical point or stationary point of f if $f_x(a,b)=0$, and $f_y(a,b)=0$, or if one of These partial derivatives does not exist.

Example: Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Find the critical point of f. Find out whether this point is a local Max or a local Min.

First we set the partial derivatives equal to zero

$f_x = 2x - 2 = 0 \Rightarrow x = 1$

$f_y = 2y - 6 = 0 \Rightarrow y = 3$

So $(1,3)$ is a critical point of the surface.

By completing the squares we will have:

$f(x,y) = (x-1)^2 + (y-3)^2 + 4$

$(x-1)^2 \geqslant 0$ and $(y-3)^2 \geqslant 0$

$\Rightarrow f(x,y) \geqslant 4$

So the point $(1,3)$ is the absolute minimum of the function

Notice $(1)^2 + (3)^2 - 2 - 18 + 4 = 4 = f(1,3)$!

Example: Find the extreme values of $f(x,y) = y^2 - x^2$.

$f_x = -2x$, $f_y = 2y \Longrightarrow (0,0)$ is the only critical point of $f(x,y)$

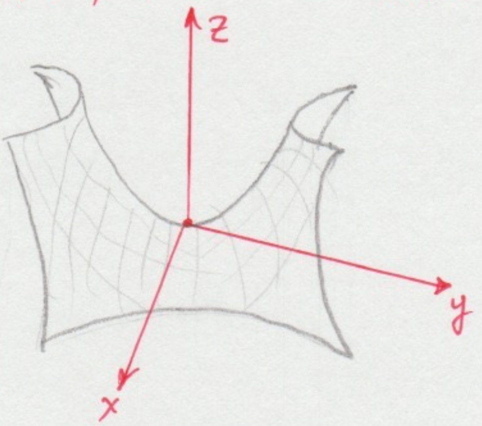
$(0,0)$ is right on the origin of the coordinate system (just the x and y values). If we set $x=0$ and move along y-axis

$f(x,y) = y^2$ so $(0,0)$ would act as the Minimum

If we set $y=0$ and move along x-axis

$f(x,y) = -x^2$ so $(0,0)$ would act as the Maximum

So $(0,0)$ is not an extreme value (Max or Min) for $f(x,y)$.



This point is called a saddle point of f.

Second Derivative Test: Suppose the second partial derivatives of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [that is, (a,b) is a critical point of f].

Let D be: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} \cdot f_{yy} - (f_{xy})^2$

a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local Minimum.

b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local Maximum.

c) If $D < 0$, Then $f(a,b)$ is not a local Max or local Min. This point is called a saddle point.

Note: If $D=0$, the test gives no information. f could have a local Min, or a local Max at (a,b) or (a,b) could be a saddle point.

Example: Find the local Max and Min values and saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

Let's find the critical points of f:

$$f_x = 4x^3 - 4y \ , \ f_y = 4y^3 - 4x \Rightarrow \begin{cases} 4x^3 - 4y = 0 \Rightarrow x^3 - y = 0 \\ 4y^3 - 4x = 0 \Rightarrow y^3 - x = 0 \end{cases}$$

$$x^3 = y \Rightarrow (x^3)^3 - x = 0$$

$$\Rightarrow x^9 - x = 0 \Rightarrow x(x^8 - 1) = x(x^4+1)(x^2-1)(x^2+1) = 0$$

$$\Rightarrow \left.\begin{matrix} x=0 \\ x=1 \\ x=-1 \end{matrix}\right\}$$ are the roots of this eq.
so the critical points are $(0,0)$, $(1,1)$, $(-1,-1)$

Now, let's calculate the second partial derivatives to construct D.

$$f_{xx} = 12x^2 \ , \ f_{yy} = 12y^2 \ , \ f_{xy} = f_{yx} = -4$$

$$D = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} \Rightarrow D = 144x^2y^2 - 16$$

for $f(0,0)$: $D = 144(0) - 16 = -16 < 0$; $(0,0)$ is the saddle point by the second derivative test.

for $f(1,1)$: $D = 144(1) - 16 > 0$, $f_{xx} = 12x^2 = 12 > 0$
so $(1,1)$ is a minimum of $f(x,y)$

for $f(-1,-1)$ $D = 144(1) - 16 > 0$, $f_{xx} = 12(-1)^2 > 0$ Therefore,
$(-1,-1)$ is also a local Minimum.

Example: Find the shortest distance from the point $(1,0,-2)$ to the plane $x+2y+z=4$.

The distance between $(1,0,-2)$ and an arbitrary point on the plane (x,y,z) is calculated as $d=\sqrt{(x-1)^2+(y-0)^2+(z+2)^2}$

$\Rightarrow d^2=(x-1)^2+y^2+(z+2)^2$, we know that z is on the plane w/ eq. $x+2y+z=4 \Rightarrow z=4-x-2y$

$\Rightarrow d^2=(x-1)^2+y^2+(4-x-2y+2)^2$

Now, Let's find the values of x, and y that minimizes d^2:

$f_x=2(x-1)+2(-1)(4-x-2y+2)=2x-2+2x+4y-12=4x+4y-14$

$f_y=2y+2(-2)(6-x-2y)=2y-24+4x+8y=10y+4x-24$

set f_x and f_y to zero $\begin{cases} 4x+4y=14 \\ 4x+10y=24 \end{cases} \Rightarrow 6y=10 \Rightarrow \boxed{y=5/3}$

$4x+\frac{20}{3}=\frac{14\times3}{3} \Rightarrow 4x=\frac{22}{3} \Rightarrow \boxed{x=\frac{11}{6}}$

so $(\frac{11}{6},\frac{5}{3})$ is the critical point of this function.

$f_{xx}=4$, $f_{yy}=10$, $f_{xy}=f_{yx}=4 \Rightarrow D=\begin{vmatrix} 4 & 4 \\ 4 & 10 \end{vmatrix}=40-16=24>0$

$D>0$, $f_{xx}>0 \Rightarrow (\frac{11}{6},\frac{5}{3})$ is a minimum for $f(x,y)=d^2$ by the second derivative test.

Example: A rectangular box w/o a lid is to be made from $12\ m^2$ of cardboard. Find The Maximum Volume of such box.

$$\text{Area} = 2xz + 2yz + \underbrace{xy}_{\text{Box does Not have a lid.}} = 12 \Rightarrow z(2x+2y) = 12 - xy \Rightarrow z = \frac{12-xy}{2(x+y)}$$

$$\text{Volume} = xyz = xy\left(\frac{12-xy}{2(x+y)}\right) = \frac{12xy - x^2y^2}{2(x+y)}$$

$$\frac{\partial V}{\partial x} = V_x = \frac{y^2(12-2xy-x^2)}{2(x+y)^2}, \quad \frac{\partial V}{\partial y} = V_y = \frac{x^2(12-2xy-y^2)}{2(x+y)^2}$$

$V_x = V_y = 0$ (we set these equal to zero to find the critical point of V)

$x = y = 0$ is one solution but it gives Volume $= \emptyset$ so we reject it.

$$\left.\begin{array}{l} 12 - 2xy - y^2 = 0 \\ 12 - 2xy - x^2 = 0 \end{array}\right\} \quad x = y \text{ is a solution of these eqs.}$$

$$\Rightarrow 12 - 2x(x) - x^2 = 0 \Rightarrow 12 - 3x^2 = 0 \Rightarrow x^2 = 4, \ x = \pm 2$$

but we reject $x = -2$ (length can not be negative)

$$\Rightarrow x = 2, y = 2 \text{ or } \boxed{x = 2, \ y = 2, \ z = 1}$$

$$z = \frac{12 - 2\times 2}{2(2+2)}$$

$$z = \frac{8}{8} = 1$$

We can use the second derivative test to prove that this point is the Max value of V

In Math 2A we learned that in order to calculate the absolute Max and Min of a function, first we find the local Max and the local Min values of a function and then we check the values of the function at the boundaries of the given interval $[a, b]$. Then we compare $f(a)$, $f(b)$ and all the local Max and local Mins to find the absolute values (Absolute Min and absolute Max values) of the function.

The process of finding Absolute Max and Absolute Min of functions of two Variables is very similar:

* To find the absolute Max and Min values of a continuous function f on a closed, bounded set D:

1- Find the values of f at the critical points of f in D (Local Max & local Min)

2- Find the extreme values of f on the boundry of D

3- The largest of the values from step 1/ and 2/ is the absolute Maximum and the smallest of these values is the absolute minimum value.

Example: Find the absolute Max and Min values of function $f(x,y) = x^2 - 2xy + 2y$ on rectangle $D = \{(x,y) \mid 0 \leq x \leq 3,\ 0 \leq y \leq 2\}$.

step 1: f is a polynomial so it is continuous on rectangle D. (it is continuous on $\mathbb{R}^2$)

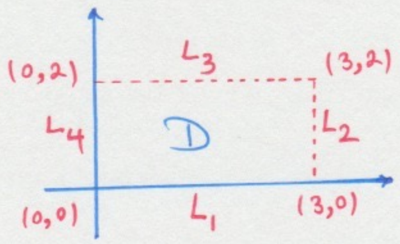
$f_x = 2x - 2y = 0 \Rightarrow y = 1$

$f_y = -2x + 2 = 0 \Rightarrow x = 1$

$\Rightarrow (1,1)$ is a critical point of f.

$f(1,1) = 1^2 - 2(1)(1) + 2(1) = 1$

step 2: Now, let's look at the values of f on the boundary of D:



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on L_1 we have $y=\emptyset$, $f(x,0)=x^2$, $0 \leq x \leq 3$

$\Rightarrow$ $f(x,0)$ increases as we go from $x=0$ to $x=3$, $f(3,0)=9$

on L_2 we have $x=3$, $f(3,y)=9-6y+2y=9-4y$, $0 \leq y \leq 2$

$f(3,y)$ decreases as we go from 0 to 2, $f(3,2)=1$

on L_3 we have $y=2$, $f(x,2)=x^2-4x+2$, $0 \leq x \leq 3$

$f(x,2)=(x-2)^2$ so the Min value on this boundary is $f(2,2)=0$ and its Max Value is $f(0,2)=4$.

on L_4 we have $x=0$, $0 \leq y \leq 2$, $f(0,y)=2y$ with the max value being $f(0,2)=4$ and the minimum value on L_4 boundary of $f(0,0)=0$.

step 3/: We compare the values in steps 1/ and 2/

critical point: $f(1,1)=1$

$f(3,0)=9$

$f(3,2)=1$

$f(2,2)=0$

$f(0,2)=4$

$f(0,0)=0$

Conclusion: $f(0,0)=f(2,2)=0$: the absolute Minima

$f(3,0)=9$: The absolute Maximum.