The Method of Lagrange Multiplies:

To find The maximum and minimum values of $f(x,y,z)$ subject to constraint $g(x,y,z)=k$ (assuming that These values exist and $\nabla g \neq 0$ on The surface $g(x,y,z)=k$:

a) Find all values of x, y, z, and λ such that:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and

$$g(x,y,z) = k$$

That is to solve The following system of equations to find four unknowns x, y, z, and λ:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = k \end{cases}$$

b) Evaluate f at all The points (x,y,z) that result from step (a). The Largest of These values is the maximum value of f; The smallest is The minimum value of f.

Method of Lagrange Multipliers:

To find Max and Min values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$, assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x,y,z)=k$:

(a) Find all the values of x, y, z, and λ such that $\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases}$

(b) Evaluate f at all points x, y, z that result from step (a). The largest of these values is the Max of f; the smallest is the minimum value of f.

In other words, we will have these equations to solve to find x, y, z, and λ.

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$f_z = \lambda g_z$$
$$g(x,y,z) = k$$

Ex. A rectangular box w/o lid is to be made from $12\,m^2$ of cardboard. Find, the maximum volume of such box. $V = xyz$ $\quad g(x,y,z) = 2xz + 2yz + xy = 12$

$$\nabla V = \lambda \nabla g$$

$$V_x = \lambda g_x \Rightarrow yz = (2z+y)\lambda \xrightarrow{\times x} xyz = \lambda(2xz + xy) \quad ①$$

$$V_y = \lambda g_y \Rightarrow xz = (2z+x)\lambda \xrightarrow{\times y} xyz = \lambda(2yz + xy) \quad ②$$

$$V_z = \lambda g_z \Rightarrow xy = (2x+2y)\lambda \xrightarrow{\times z} xyz = \lambda(2xz + 2yz) \quad ③$$

$$2xz + 2yz + xy = 12$$

from ① and ② $\quad 2xz + \cancel{xy} = 2yz + \cancel{xy}$

$$\Rightarrow \boxed{x = y}$$

from ②, ③ $\rightarrow \quad 2\cancel{y}z + xy = 2\cancel{y}z + 2xz \Rightarrow \boxed{y = 2z}$

$$\Rightarrow 2xz + 2yz + xy = 12 \Rightarrow 2(2z)z + 2(2z)z + (2z)(2z) = 12$$

$$\Rightarrow 4z^2 + 4z^2 + 4z^2 = 12 \Rightarrow z = \pm 1$$

we reject $z=-1$ $\Rightarrow$ $z=1$, $y=2$, $x=2$

Example: Find the extreme values of $f(x,y)=x^2+2y^2$ on the circle $x^2+y^2=1$.

We are asked to find extreme values of f subject to constraint $x^2+y^2=1$.

$\nabla f = \lambda \nabla g$

$\Rightarrow$ $2x=\lambda \cdot 2x \Rightarrow x=0$ or $\lambda=1$, if $x=0 \Rightarrow y=\pm 1$

$4y=\lambda(2y)$ if $\lambda=1 \Rightarrow y=0 \Rightarrow x=\pm 1$

Therefore, f has 4 possible extreme values at $(0,+1)$, $(0,-1)$, $(+1,0)$, $(-1,0)$

$f(0,1)=2$ so, f has two max values at $(0,\pm 1)$

$f(0,-1)=2$ and two Min values at $(\pm 1, 0)$

$f(1,0)=1$

$f(-1,0)=1$

Example: Find extreme values of $f(x,y)=x^2+2y^2$ on the disk $x^2+y^2\leq 1$

First, we try to find the critical points of f.

$f_x=2x=0 \Rightarrow x=0$
$f_y=4y=0 \Rightarrow y=0$ $\Rightarrow$ $(0,0)$ is a critical point.

$\Rightarrow f(0,0)=0$

Also, from the previous example, we have: $f(\pm 1,0)=1$, $f(0,\pm 1)=2$

(remember, we have to check the boundaries for absolute min & max calculations)

$\Rightarrow$ Max value of f on the disk $x^2+y^2\leq 1$ is $f(0,\pm 1)=2$ and the minimum value is $f(0,0)=0$.

Example: Find the point on the sphere $x^2+y^2+z^2=4$ that are closest to and the farthest from the point $(3,1,-1)$.

The distance from an arbitrary point (x,y,z) on the surface of sphere and point $(3,1,-1)$ is

$$f(x,y,z)=d=\sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$$

$$\Rightarrow d^2=(x-3)^2+(y-1)^2+(z+1)^2$$

Now, we want to see what are the values of (x,y,z) that maximizes and minimizes d^2 given the constraint $x^2+y^2+z^2=4$.

$$g(x,y,z)=x^2+y^2+z^2=4$$

$$\nabla f=\lambda\nabla g \Rightarrow \langle 2(x-3), 2(y-1), 2(z+1)\rangle=\lambda\langle 2x, 2y, 2z\rangle$$

$$\Rightarrow \left.\begin{array}{l} 2\lambda x=2x-6 \longrightarrow \lambda x=x-3 \\ 2\lambda y=2y-2 \longrightarrow \lambda y=y-1 \\ 2\lambda z=2z+2 \longrightarrow \lambda z=z+1 \end{array}\right\} \begin{array}{l} \rightarrow x=\frac{3}{1-\lambda} \\ \rightarrow y=\frac{1}{1-\lambda} \\ \rightarrow z=\frac{-1}{1-\lambda} \end{array}$$

$$\Rightarrow \frac{9}{(1-\lambda)^2}+\frac{1}{(1-\lambda)^2}+\frac{1}{(1-\lambda)^2}=4 \Rightarrow \frac{11}{(1-\lambda)^2}=4 \Rightarrow (1-\lambda)^2=\frac{11}{4}$$

$$\Rightarrow 1-\lambda=\pm\frac{\sqrt{11}}{2} \Rightarrow \lambda=\begin{cases} 1-\frac{\sqrt{11}}{2} \\ 1+\frac{\sqrt{11}}{2} \end{cases}$$

if we use these two values of λ to find x,y,z, we get the following two points:

$$\underbrace{\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)}_{\text{Smaller value if we plug in the coordinates in } f(x,y,z)} \text{ and } \underbrace{\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)}_{\text{Larger value if we plug in the coordinates in } f(x,y,z)}$$

Two Constraints:

Suppose that we want to find the Max and Min values of a function $f(x,y,z)$ subject to two constraints of $g(x,y,z) = k$, and $h(x,y,z) = c$. The process is very similar to the case w/ one constraint, except we have two Lagrange multipliers, λ, and μ.

$$\Rightarrow \boxed{\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)}$$

Example: Find the Max value of the function $f(x,y,z) = x + 2y + 3z$ on the curve of intersection of plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

$f(x,y,z) = x + 2y + 3z$

$g(x,y,z) = x - y + z = 1$

$h(x,y,z) = x^2 + y^2 = 1$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$\Rightarrow f_x:\ 1 = \lambda(1) + \mu(2x) \Rightarrow -2 = 2\mu x \rightarrow x = -\frac{1}{\mu}$$

$$f_y:\ 2 = \lambda(-1) + \mu(2y) \Rightarrow 5 = 2\mu y \rightarrow y = \frac{5}{2\mu}$$

$$f_z:\ 3 = \lambda(1) + \mu(0) \Rightarrow \lambda = 3$$

$$x - y + z = 1$$

$$x^2 + y^2 = 1$$

$$x^2 + y^2 = 1 \Rightarrow \frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1 \Rightarrow \frac{1}{\mu^2}\left(\frac{29}{4}\right) = 1 \Rightarrow \mu = \pm\frac{\sqrt{29}}{2}$$

$$\Rightarrow x = \pm\frac{2}{\sqrt{29}} \qquad y = \pm\frac{5}{\sqrt{29}}, \qquad z = 1 \pm \frac{7}{\sqrt{29}}$$

The corresponding points are $P_1\left(\frac{-2}{\sqrt{29}}, \frac{+5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}\right) \rightarrow f(P_1) = 3 + \sqrt{29}$

$P_2\left(+\frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}\right) \rightarrow f(P_2) = 3 - \sqrt{29}$

Therefore, the Max value of f occurs at $\left(\frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}\right)$ and is $3 + \sqrt{29}$.