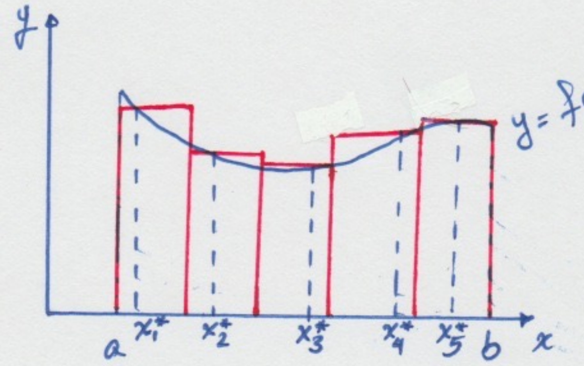
Introduction:

Similar to what we did in Math 2B (our attempt to solve the area problem led to the definition of a definite integral), we try to find The Volume of a Solid and we will see that in the process, we will arrive at the definition of a double integral.

From Math 2B, you remember that in order to find the area under the graph of $y=f(x)$ when $a \leq x \leq b$, we break the area to rectangles as followed:



$$\Delta x = \frac{b-a}{n}$$

in This case $n=5$

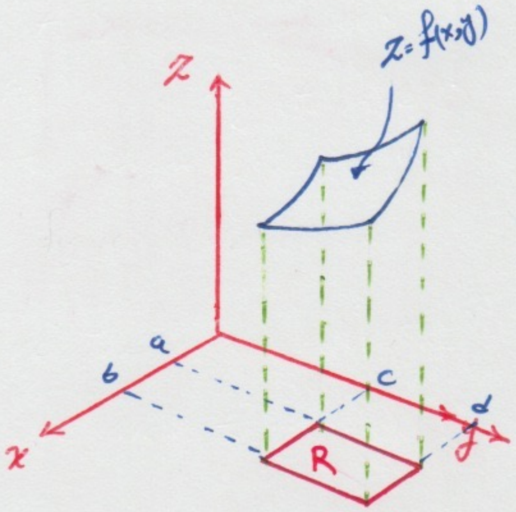
$$A \approx \sum_{i=1}^{n=5} f(x_i^*) \Delta x$$

We called This Riemann Sum. Then we defined concept of definite integrals as the limit of Riemann Sum when n approaches infinity:

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

In a very Similar way, we can show that The Volume of a geometric shape may be shown as a double integral (or approximately as a double Riemann Sum).

Volumes and double integrals:

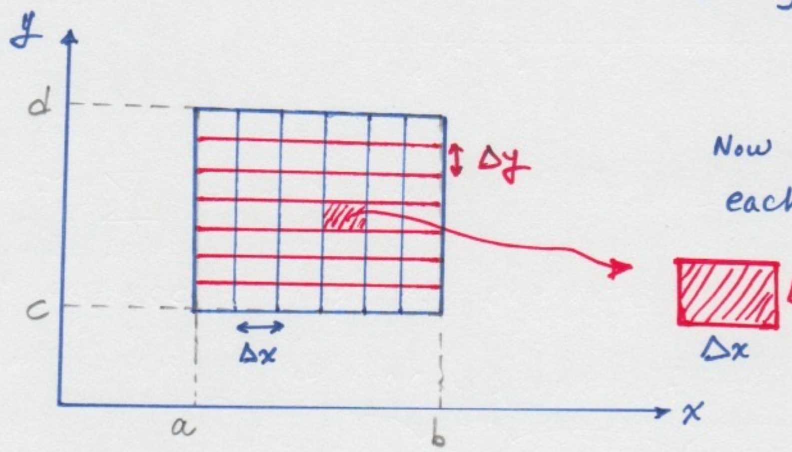


Now, suppose that f is a function of Two variables and is a graph of a surface with equation $z = f(x, y)$

Let's assume that f is defined in a rectangular region where $a \leq x \leq b$ and $c \leq y \leq d$

we are interested in finding the volume of Solid S which is the solid that lies above rectangular region R and under the graph of f.

our first task is to divide the rectangular region into subrectangles. we divide $[a,b]$ into m subintervals and $[c,d]$ into n subintervals.



$$\Delta x = \frac{b-a}{m} \qquad \Delta y = \frac{d-c}{n}$$

Now it is obvious that the area of each subrectangle is equal to $\Delta x \Delta y$

$$\Delta A = \Delta x \Delta y$$

Notice that we have i and j here

Now if we Pick an arbitrary Point inside this subrectangle (say (x^*_{ij}, y^*_{ij}))

The height of the rectangular cube multiplied by the area ΔA gives us the volume of one of the rectangular cubes.

(Look at the figures in the next page)

Figure 4
chapter 15.1
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Figure 5
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Sorry, the publisher's webpage was not accessable so I could not copy paste these two figures. will fix them soon.

In other words: $\Delta V = f(x^*_{ij}, y^*_{ij})\,\Delta A$

we follow the same procedure for all the rectangles and add the volume of the corresponding boxes to get an approximation of the total volume of Solid S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\,\Delta A$$

This is called double Riemann Sum

$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\,\Delta A$$

Important definition:

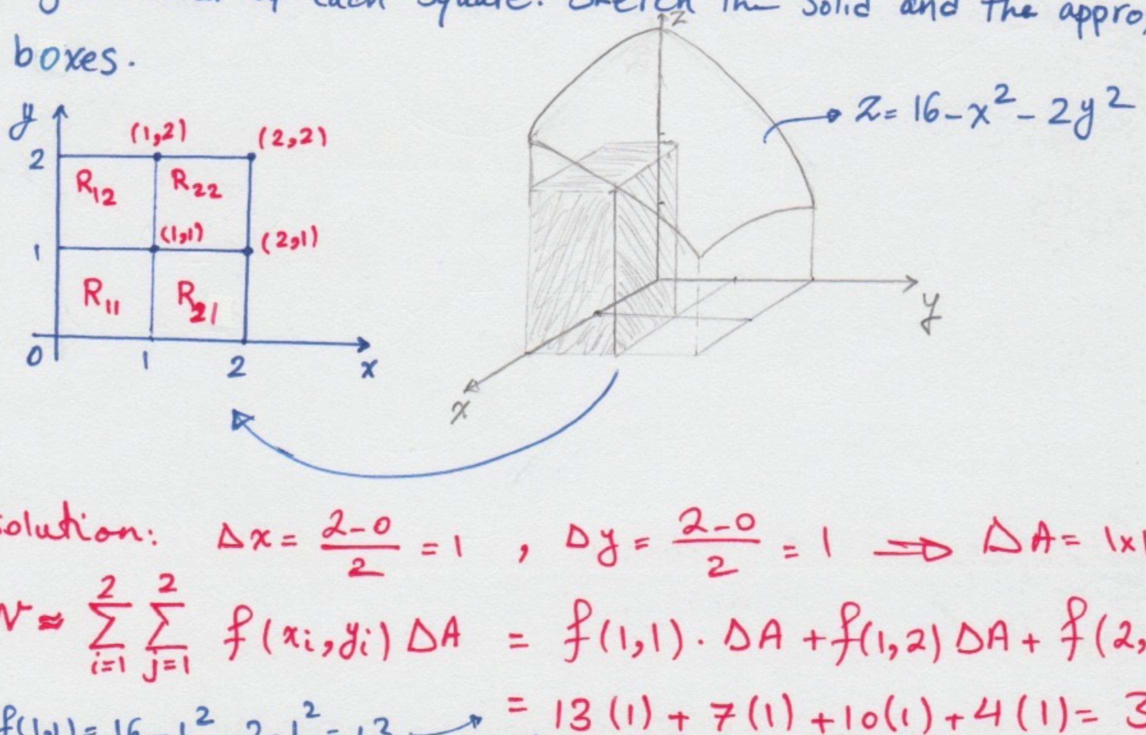
The double integral of f over the rectangular region R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

if the limit exists.

if $f(x,y) \geqslant 0$ then the volume V of the solid that lies above rectangle R and below the surface $z = f(x,y)$ is $V = \iint_R f(x,y)\,dA$

Example: Estimate the volume of the solid that lies above the square $R = [0,2]\times[0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. divide R into four equal squares and choose the sample point to be the upper right corner of each square. Sketch the solid and the approximating rectangular boxes.



solution: $\Delta x = \frac{2-0}{2} = 1$, $\Delta y = \frac{2-0}{2} = 1 \Rightarrow \Delta A = 1\times 1 = 1$

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_i)\,\Delta A = f(1,1)\cdot\Delta A + f(1,2)\,\Delta A + f(2,1)\,\Delta A + f(2,2)\,\Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$ → this is the approximate volume of the solid.

$f(1,1) = 16 - 1^2 - 2\cdot 1^2 = 13$

$f(1,2) = 16 - 1^2 - 2\cdot 2^2 = 7$

$f(2,2) = 16 - 2^2 - 2\times 2^2 = 4$

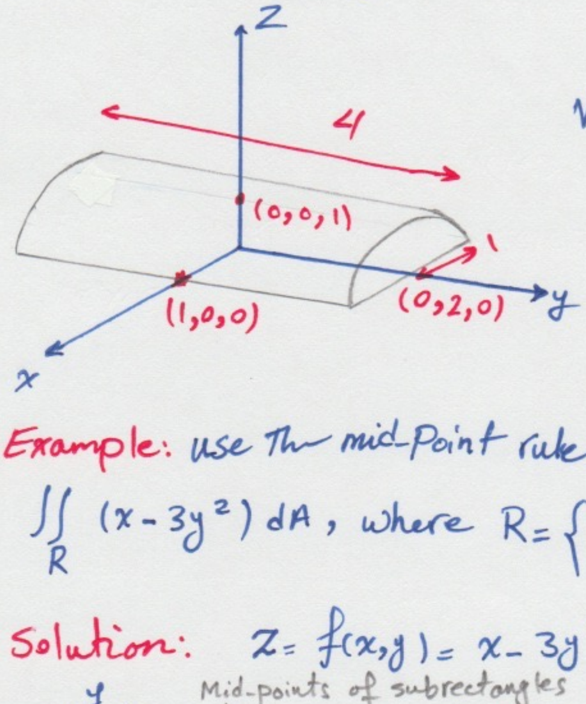
$f(2,1) = 16 - 2^2 - 2\cdot 1^2 = 10$

Example: If $R = \{(x,y) \mid -1 \leq x \leq 1,\ -2 \leq y \leq 2\}$, evaluate the integral $\iint_R \sqrt{1-x^2}\, dA$?

Solution: it is not easy to evaluate this integral using the definition. However, we can interpret this double integral as a Volume (because $f(x,y)$ is positive).

$$f(x,y) = z = \sqrt{1-x^2} \Rightarrow x^2 + z^2 = 1$$
$$z \geq 0$$

Therefore, the given double integral is the volume of the solid that lies below the circular cylinder $x^2+z^2=1$ and above the rectangular region R in xy plane.



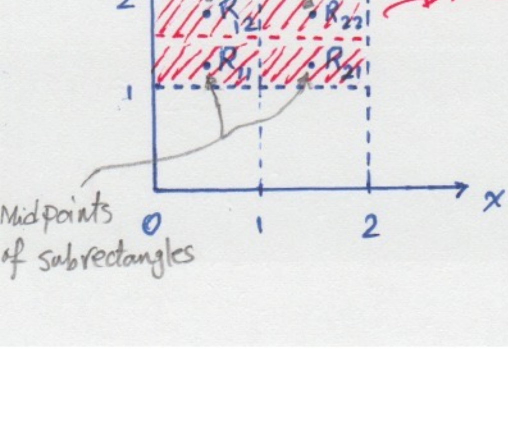
volume of the half Cylinder $= \frac{1}{2} \cdot h \cdot \pi r^2$

$$V = \pi \times 1^2 \times \frac{1}{2} \times 4 = 2\pi$$

$$\Rightarrow V = 2\pi$$

Example: use the mid-Point rule w/ $m=n=2$ to estimate the value of the integral $\iint_R (x-3y^2)\, dA$, where $R = \{(x,y) \mid 0 \leq x \leq 2,\ 1 \leq y \leq 2\}$.

Solution: $z = f(x,y) = x - 3y^2$



$$\iint_R (x-3y^2)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j) \Delta A$$

$$= f(\tfrac{1}{2}, \tfrac{5}{4}) \cdot \Delta A + f(\tfrac{3}{2}, \tfrac{5}{4}) \Delta A + f(\tfrac{1}{2}, \tfrac{7}{4}) \Delta A + f(\tfrac{3}{2}, \tfrac{7}{4}) \Delta A$$

$$= \frac{1}{2}\left(-\frac{67}{16} - \frac{139}{16} - \frac{51}{16} - \frac{123}{16}\right) = -\frac{95}{8}$$

$$\Rightarrow \boxed{\iint_R (x-3y^2)\, dA = -11.875}$$

Iterated Integrals:

From math 2B, you remember that evaluating single integrals using the def. of an integral is usually difficult. Doing so for double integrals is even more difficult. However, the same way that the Fundamental Theorem of Calculus provided a much easier method to evaluate single integrals, we can express a double integral as an iterated integral and use FTC for each iteration. (This Process is Called Partial Integration)

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \Big[\underbrace{\int_c^d f(x,y)\,dy}_{\text{We evaluate this integral first.}} \Big]\,dx$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \Big[\underbrace{\int_a^b f(x,y)\,dx}_{\text{We evaluate this integral first.}} \Big]\,dy$$

Example: Evaluate the following iterated integrals.

a) $\int_0^3 \int_1^2 x^2 y\,dy\,dx$ b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy$

$$= \int_0^3 \left[\int_1^2 x^2 y\,dy \right] dx = \int_0^3 x^2 \left[\int_1^2 y\,dy \right] dx$$

why? because for the inner integral, x^2 is a constant and comes out of integral

$$= \int_0^3 x^2 \left[\frac{y^2}{2} \Big|_1^2 \right] dx = \int_0^3 x^2 \left(\frac{2^2}{2} - \frac{1^2}{2} \right) dx$$

$$= \int_0^3 \frac{3}{2} x^2 \cdot dx = \frac{3}{2} \cdot \frac{x^3}{3} \Big]_{x=0}^{x=3} = \frac{1}{2}\left(3^3 - 0^3\right) = \boxed{\frac{27}{2}}$$

b) $$\int_1^2 \int_0^3 x^2 y\,dx \cdot dy = \int_1^2 y \left[\int_0^3 x^2 \cdot dx \right] dy = \int_1^2 y \left[\frac{x^3}{3} \right]_{x=0}^{x=3} \cdot dy = \int_1^2 y \left(\frac{27}{3} - \frac{0}{3} \right) dy$$

$$= 9 \int_1^2 y\,dy = 9 \cdot \frac{y^2}{2} \Big]_{y=1}^{y=2} = 9 \cdot \left(\frac{4}{2} - \frac{1}{2} \right) = 9 \cdot \frac{3}{2} = \boxed{\frac{27}{2}}$$

As it was seen in the previous example, in iterated integrals, the order of integration does not matter.

Fubini's Theorem: If f is continuous on the rectangle:
$R = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Example: Evaluate the double integral $\iint_R (x - 3y^2)\,dA$, where $R = \{(x,y) \mid 0 \leq x \leq 2,\ 1 \leq y \leq 2\}$ (Compare your results w/ the example in page 5)

Solution: According to Fubini's Theorem:

$$\iint_R (x-3y^2)\,dA = \underbrace{\int_0^2 \int_1^2 (x-3y^2)\,dy\,dx}_{1} = \underbrace{\int_1^2 \int_0^2 (x-3y^2)\,dx \cdot dy}_{2}$$

1: $$\int_0^2 \Big[xy - y^3 \Big]_{y=1}^{y=2} \cdot dx = \int_0^2 \Big[(2x-8)-(x-1)\Big]\,dx = \int_0^2 (x-7)\,dx$$

$$= \frac{x^2}{2} - 7x \Big]_{x=0}^{x=2} = \left(\frac{4}{2} - 7\times 2\right) - (0-0) = 2 - 14 = \boxed{-12}$$

2: $$\int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} \cdot dy = \int_1^2 \left[\left(\frac{4}{2} - 3\times 2y^2\right) - (0-0)\right] dy = \int_1^2 (2-6y^2)\,dy$$

$$= 2y - 2y^3 \Big]_1^2 = (2\times 2 - 2\times 2^3) - (2-2) = 4 - 16 = \boxed{-12}$$

Example: Evaluate $\iint_R y\sin(xy)\,dA$ where $R=[1,2]\times[0,\pi]$.

Solution: According to Fubini's Theorem:

$$\iint_R y\,\sin(xy)\,dA = \int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy$$

$$= \int_0^{\pi} -\cos(xy)\Big]_{x=1}^{x=2} \cdot dy = -\int_0^{\pi}(\cos 2y - \cos y)\,dy$$

$$= -\left(\frac{1}{2}\sin 2y - \sin y\right)\Big]_0^{\pi} = \sin y - \frac{1}{2}\sin 2y\Big]_{y=0}^{y=\pi}$$

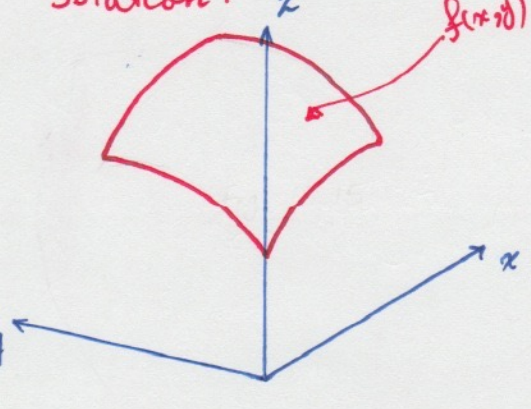
$$= \left(\sin\pi - \frac{1}{2}\sin 2\pi\right) - \left(\sin 0 - \frac{1}{2}\sin 0\right) = \boxed{0}$$

Final Ans.

very important note:
If we reverse the order of integration and try to evaluate $\int_1^2\int_0^{\pi} y\,\sin(xy)\,dy\,dx$, we should get the same result. However, evaluation of this recent double integral is much more difficult that the solution presented above. Therefore, it is wise to choose the order of integration such that it leads to simpler integrals.

Example: Find the volume of the solid that is bounded by elliptic paraboloid $x^2+2y^2+z=16$, the plane $x=2$ and $y=2$ and the three coordinate planes.

solution:



We can use Fubini's theorem and solve for the volume of the solid.

$$V=\iint_R (16-x^2-2y^2)\,dA = \int_0^2\int_0^2 (16-x^2-2y^2)\,dx\,dy$$

$$= \int_0^2 \left[\left(16x-\frac{x^3}{3}-2xy^2\right)\right]_{x=0}^{x=2} dy = \int_0^2\left(\frac{88}{3}-4y^2\right)dy$$

$$= \left[\frac{88}{3}y - \frac{4}{3}y^3\right]_0^2 = \boxed{48}$$

A Special Case (Important):

In the special case where $f(x,y)$ can be factored as the product of a function of x only, and a function of y only: $f(x,y) = g(x) \cdot h(y)$

Then: *

$$\boxed{\iint_R f(x,y)\,dA = \iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy}$$

where $R = [a,b] \times [c,d]$ ⇝ again this means $a \le x \le b$, $c \le y \le d$

Example: If $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, Then, Evaluate $\iint_R \sin x \cos y \, dA$.

Solution: If $0 \le x \le \frac{\pi}{2}$ and $0 \le y \le \frac{\pi}{2}$

$$\iint_R \sin x \cos y \, dA = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin x \cos y \, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \sin x \, dx \cdot \int_0^{\frac{\pi}{2}} \cos y \, dy$$

$$= -\cos x \Big]_{x=0}^{x=\frac{\pi}{2}} \cdot \sin y \Big]_0^{\pi/2}$$

$$= \Big[-(\underbrace{\cos \tfrac{\pi}{2}}_{0} - \underbrace{\cos 0}_{1})\Big]\Big[\underbrace{\sin \tfrac{\pi}{2}}_{1} - \underbrace{\sin 0}_{0}\Big] = -(-1)\cdot(1)$$

$$\boxed{= 1}$$ Final Ans.

Average Value

In Math 2B (if you remember) we had a concept called average value. The average value of a function f of one variable defined on an interval $[a,b]$ is $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

In the same fashion, we may define the average value of a function f of two variables: $f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$

Area of region R