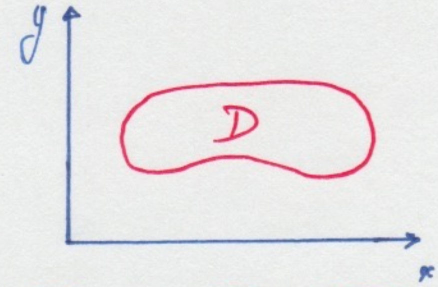
Double Integrals over general regions

section 15.2

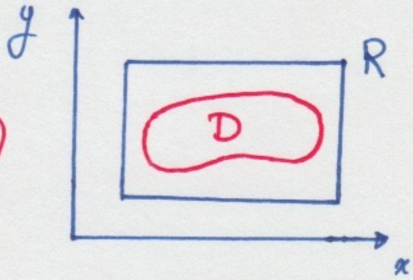
when dealing with single integrals, the region that we integrete is always an interval. In the previous section, we learned to deal with double integrals over rectangular regions. Now we want to learn double integration over a general bounded region, D.



we have $f(x,y)$ and we want to integrate over region D.

To do so, we define a new function $F(x,y)$ with the rectangular domain R.

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ in in} D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but Not } D \end{cases}$$



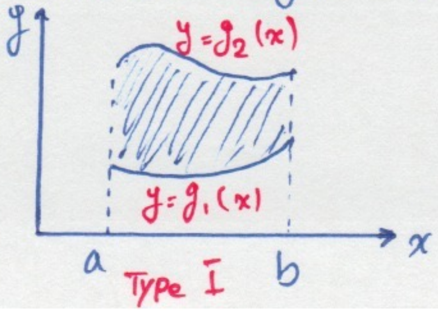
if F is integrable over R, Then we define the double integral of $f(x,y)$ over D by:

$$\boxed{\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA}$$

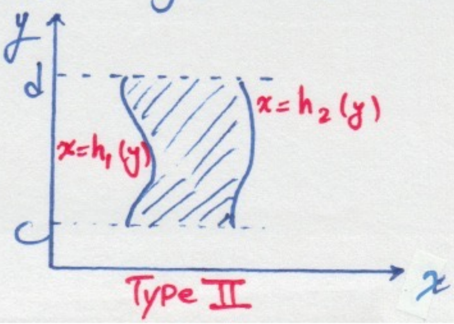
Notice the difference b/w these Two functions.

Type I & Type II Region:

when evaluating double integrals, it is often helpful to Recognize Two different types of the Region that we might deal with.



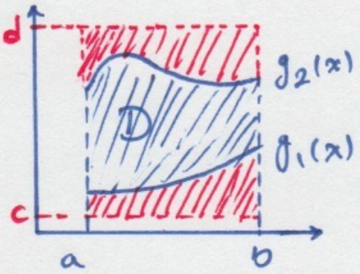
Type I



Type II

Type I Region is the region that lies between the graphs of two continuous functions of x: $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Now, if we go ahead and define a rectangular region around D, we will have:



Combining the definition that we just learned in the previous page w/ Fubini's Theorem:

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

We can easily observe for $g_1(x) \le y \le g_2(x)$, $f(x,y) = F(x,y)$ and for values of $y < g_1(x)$ or $y > g_2(x)$, $F(x,y) = 0$. This means that outside the range $g_1(x) \le y < g_2(x)$, $F(x,y)$ does <u>NOT</u> have any contribution to the integral. so, instead of $\underline{\underline{c}}$ and $\underline{\underline{d}}$ (the integration bounds) we can just go ahead and use $g_1(x)$ and $g_2(x)$:

* If f is continuous on a type I region D such that

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\} \Rightarrow \iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

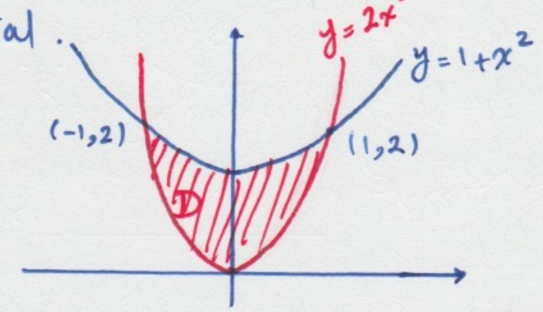
Similarly, for a type II region we will have:

If f is continuous on a type II region D such that

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\} \Rightarrow \iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example: Evaluate $\iint_D (x+2y)\,dA$ where D is the region bounded by the Parabolas $y=2x^2$ and $y=1+x^2$.

Solution: First, let's understand region D before we even try to evaluate The integral.



to find The intersection points:

$$2x^2 = 1+x^2 \Rightarrow x = \pm 1$$

$$x=1 \rightarrow y=+2$$

$$x=-1 \rightarrow y=+2$$

We notice that The region D is a type I region.

$$D=\{(x,y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1+x^2\}$$

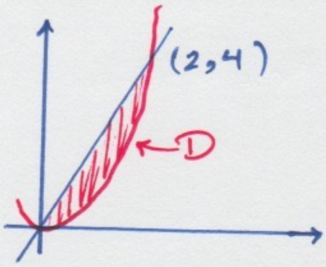
$$\Rightarrow \iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx = \int_{-1}^{1} \left[xy+y^2\right]_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} \left[x(1+x^2)+(1+x^2)^2 - x(2x^2)-(2x^2)^2\right] dx = \int_{-1}^{1}\left(-3x^4-x^3+2x^2+x+1\right) dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x \Big]_{-1}^{1} = \frac{32}{15}$$

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and parabola $y = x^2$.

Solution: If we plot the region enclosed between the line $y = 2x$ and the parabola $y = x^2$ we will have



$2x = x^2 \Rightarrow x = 0$, $x = 2$

$$\Rightarrow D = \{(x,y) \mid 0 \leqslant x \leqslant 2,\ x^2 \leq y \leq 2x\}$$

$$\Rightarrow V = \iint_D (x^2+y^2)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx = \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{2x} dx$$

$$\Rightarrow V = \int_0^2 \left[x^2 \cdot 2x + \frac{(2x)^3}{3} - x^2(x^2) - \frac{(x^2)^3}{3}\right] dx = \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right) dx$$

$$\Rightarrow V = -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Big]_0^2 = \frac{216}{35}$$

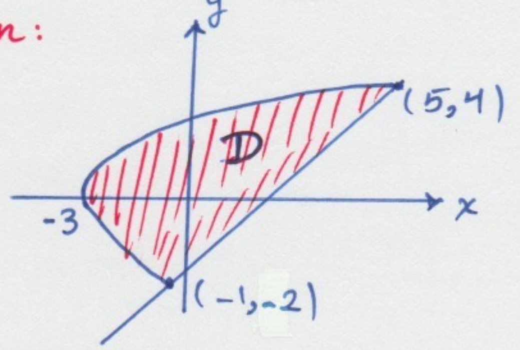
Note: We could assume that the region D is defined as

$$D = \{(x,y) \mid 0 \leq y \leq 4,\ y/2 \leq x \leq \sqrt{y}\}$$

In other words, we could treat region D as a **type II** region, an solve the problem. This would give exactly the same ans.

Example: Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.

Solution:



$$\frac{y^2-6}{2} = y+1 \Rightarrow y^2-6 = 2y+2$$

$$\Rightarrow y^2 - 2y - 8 = 0 \Rightarrow y = -2, \quad y = 4$$

$$\begin{cases} y=-2, & x=-1 \\ y=4, & x=5 \end{cases}$$

Again we can assume that this is either a type I, or type II region. However, treating the region as a type II region is easier:

$$D = \left\{ (x,y) \;\middle|\; -2 \le y \le 4, \quad \frac{y^2-6}{2} \le x \le y+1 \right\}$$

$$\Rightarrow \iint_D xy\,dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy = \int_{-2}^{4} \left(\frac{x^2}{2} y \Big]_{\frac{y^2}{2}-3}^{y+1} \right) dy$$

$$\Rightarrow = \int_{-2}^{4} \left[(y+1)^2 \cdot \frac{y}{2} - \left(\frac{y^2}{2}-3\right)^2 \cdot \frac{y}{2} \right] dy = \frac{1}{2}\int_{-2}^{4} \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y \right) dy$$

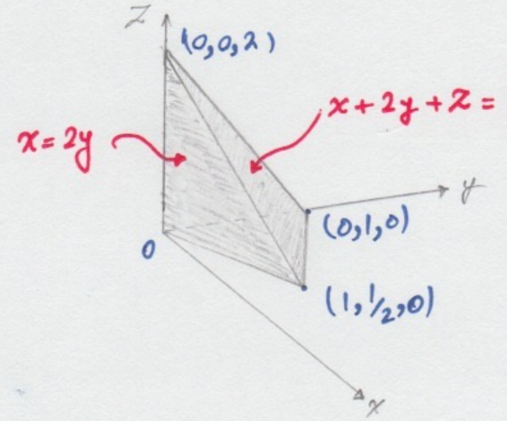
$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2 \right]_{-2}^{4} = 36$$

As was mentioned above, we would have obtained a similar result evaluating

$$\underbrace{\iint_D xy\,dA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx}_{\text{treating region } D \text{ as a type I region.}}$$

Example: Find the volume of a tetrahedron bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$, and $z=0$.

Solution: If we draw the 3D sketch of the tetrahedron, we will have:



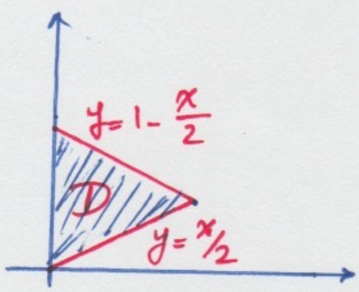
$x=2y$ is a vertical plane (perpendicular to xy plane). The intersection of these two planes is the line $x=2y$ or $y=\frac{x}{2}$

Plane $x+2y+z=2$ also intersects with xy plane ($z=0$) at line $x+2y=2$ or

$$y=-\frac{x}{2}+1$$

The other plane of the tetrahedron is $x=0$ perpendicular to xy plane. The intersection occurs along y-axis.

Therefore the region D will look like:



$$D=\left\{(x,y)\mid 0\leq x\leq 1,\ \frac{x}{2}\leq y\leq 1-\frac{x}{2}\right\}$$

$$V=\iint_D (2-x-2y)\,dA=\int_0^1\int_{\frac{x}{2}}^{1-\frac{x}{2}}(2-x-2y)\,dy\,dx$$

$$=\int_0^1\left[2y-xy-y^2\right]_{y=\frac{x}{2}}^{y=1-\frac{x}{2}}dx=\int_0^1\left[2-x-x\left(1-\frac{x}{2}\right)-\left(1-\frac{x}{2}\right)^2-x+\frac{x^2}{2}+\frac{x^2}{4}\right]dx$$

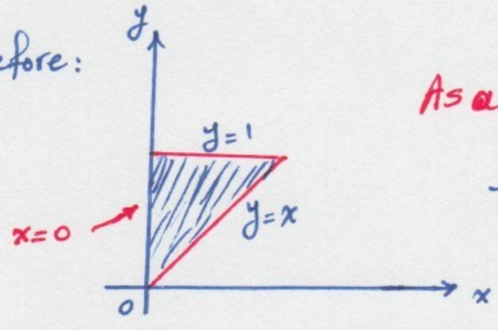
$$=\int_0^1(x^2-2x+1)\,dx=\frac{x^3}{3}-x^2+x\Big]_0^1=\frac{1}{3}$$

Example: Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

Solution: If we try to solve this integral as is, the task is imposible (in the sense that the result of integration is <u>NOT</u> an elementary function and can <u>NOT</u> be expressed in finite terms). However, we can chang the order of integration by converting region D from a type I region to a type II region.

we have $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$ which is $\int_{x=0}^{x=1} \int_{y=x}^{y=1} \sin(y^2)\,dy\,dx$.

therefore:



As a type I region: $D=\{(x,y)\mid 0\le x\le 1,\ x\le y\le 1\}$

This same region can be considered a type II region:

$$D=\{(x,y)\mid 0\le y\le 1,\ 0\le x\le y\}$$

Note: it is very important to understand the difference b/w region D defined as a type II and region D defined as a type I region. (you also need to understand how we have obtained this New definition).

Think about it this way:
- **Type I:** $y_1\le x\le y_2$, lower curve $\le y\le$ upper curve
- **Type II:** $x_1\le y\le x_2$, left curve $\le x\le$ right curve

Now we can write $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA = \int_0^1\int_0^y \sin(y^2)\,dx\,dy$

$$= \int_0^1 \left[x\sin(y^2)\right]_{x=0}^{x=y} dy = \int_0^1 y\sin(y^2)\,dy = -\tfrac{1}{2}\cos(y^2)\Big]_0^1$$

$$= \tfrac{1}{2}(1-\cos 1)$$

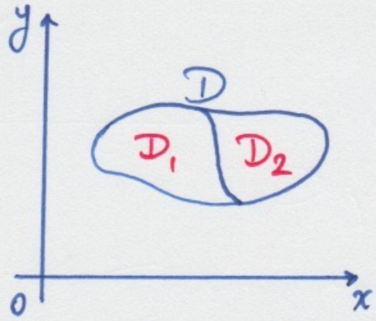
Properties of Double integrals:

* $\iint_D [f(x,y) + g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

* $\iint_D c f(x,y)\,dA = c\iint_D f(x,y)\,dA$ where c is a constant

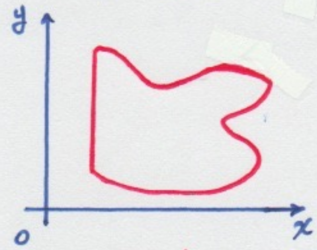
* If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then $\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$

* If $D = D_1 \cup D_2$, where D_1 and D_2 do Not overlap except perhaps on their boundaries, then:

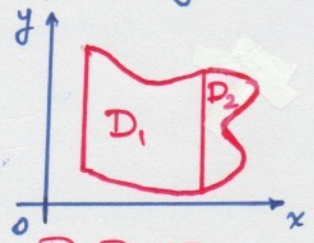


$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

Note: The above property is important when evaluating double integrals over region D that are neither type I, nor type II (see the example below)



D is neither type I Nor type II



$D = D_1 \cup D_2$

D_1 is type I, D_2 is type II

* If we integrate the constant function $f(x,y) = 1$ over a region D, we get the area of D:

$$\iint \underbrace{1}_{f(x,y)=1}\,dA = \underbrace{A(D)}_{\text{Area of region } D}$$

* If $m \leq f(x,y) \leq M$ for all (x,y) in D, then

$$\boxed{m\,A(D) \leq \iint_D f(x,y)\,dA \leq M\,A(D)}$$

Example: Estimate the integral $\iint_D e^{\sin x \cos y}\,dA$ where D is the disk w/ center the origin and radius 2.

Solution: we know $\left.\begin{array}{l} -1 \leq \sin x \leq +1 \\ -1 \leq \cos y \leq +1 \end{array}\right\} \Rightarrow -1 \leq \sin x \cos y \leq +1$

$$\Rightarrow e^{-1} \leq e^{\sin x \cos y} \leq e^{1}$$

using $m = e^{-1} = \frac{1}{e}$ and $M = e^{1} = e$ and $A(D) = \pi r^2 = \pi (2)^2$, we obtain

$$\frac{4\pi}{e} \leq \underbrace{\iint_D e^{\sin x \cos y}\,dA}_{} \leq 4\pi e$$

we estimated that the integral is between $\frac{4\pi}{e}$ and $4\pi e$