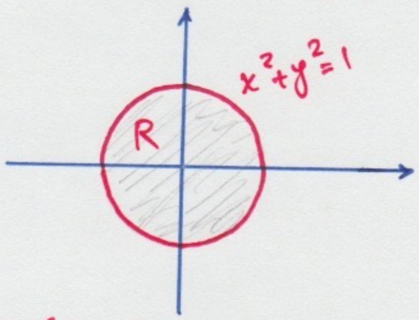
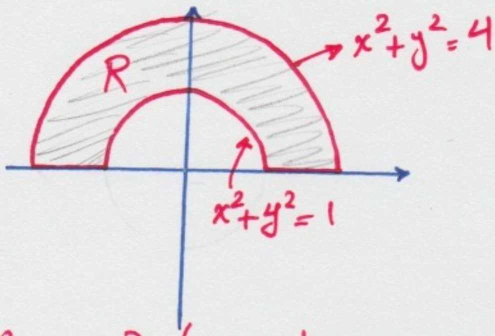
Intro: Suppose that we want to evaluate a double integral $\iint_R f(x,y)\,dA$ where R is a circular-shaped region. Describing R in terms of rectangular coordinates is rather complicated for such region. Therefore, we describe region R for such problems easily in **Polar Coordinates**.

Examples: The following circular shaped regions maybe described easily, using Polar Coordinates:





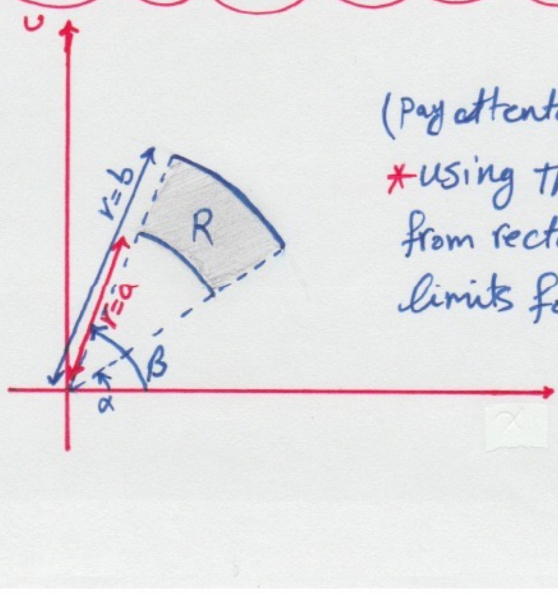
$R=\{(r,\theta)\mid 0\leq r\leq 1,\ 0\leq\theta\leq 2\pi\}$ $\qquad$ $R=\{(r,\theta)\mid 1\leq r\leq 2,\ 0\leq\theta\leq\pi\}$

Change to polar coordinates in a double integral: (very important)

If f is continuous on a polar rectangle R given by $0\leq a\leq r\leq b$, $\alpha\leq\theta\leq\beta$, where $0\leq\beta-\alpha\leq 2\pi$ ($\beta-\alpha$ is between zero and 2π), then:

$$\iint f(x,y)\,dA=\int_\alpha^\beta\int_a^b f(r\cos\theta,\, r\sin\theta)\, r.\,dr.\,d\theta$$

Be Careful, Do <u>NOT</u> forget about this <u>r</u>.



(Pay attention to angles α, β and radii a & b)

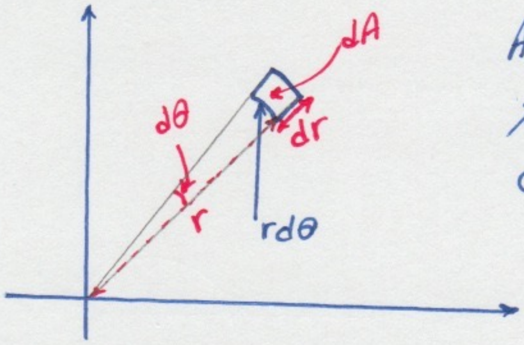
*using this formula, we can convert double integrals from rectangular to polar coordinates, using appropriate limits for r and θ and replacing dA by $r\,dr\,d\theta$.
We also need to rewrite x and y as $x=r\cos\theta$ and $y=r\sin\theta$.

Double Integrals In Polar Coordinates

Section 15.3

The proof and details of the formula for conversion of rectangular coordinates to Polar Coordinates is provided for you in page 1011 of your book. Although, the following is not a proof, it gives you a general idea why $dA = r\,dr\,d\theta$

* Suppose that we divide the region R into smaller circular subsectors:

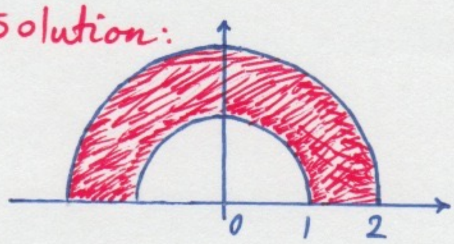


Assuming that the polar rectangle is very close to ordinary rectangle, the area of rectangle, dA can be calculated as length x width

$$\Rightarrow dA = (r\,d\theta)\cdot dr = r\,dr\cdot d\theta$$

Example: Evaluate $\iint_R (3x+4y^2)\,dA$ where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution:



The region R in the polar Coordinate is given as

$$R = \{(r,\theta) \mid 1 \le r \le 2 \text{ and } 0 \le \theta \le \pi\}$$

$$\iint_R (3x+4y^2)\,dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta = \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta\right]_{r=1}^{r=2} d\theta$$

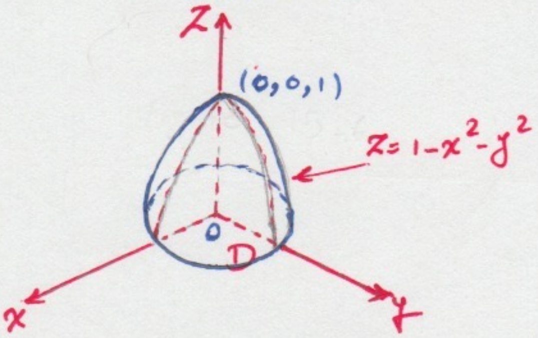
$$= \int_0^\pi (7\cos\theta + 15\sin^2\theta)\,d\theta = \int_0^\pi \left[7\cos\theta + \frac{15}{2}(1-\cos 2\theta)\right] d\theta$$

$$= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta \Big]_0^\pi = \frac{15\pi}{2}$$

Example: Find the volume of the solid bounded by the plane $z=0$ and the Paraboloid $z=1-x^2-y^2$.

Solution: paraboloid $z=1-x^2-y^2$ and plane $z=0$ intersect at

$z \leftarrow 0=1-x^2-y^2 \Rightarrow x^2+y^2=1$ (A circle). Therefore, the solid that we are interested lies above circular disk $x^2+y^2=1$ and under the Paraboloid.



In polar Coordinates, region D is given by

$$0\leq r\leq 1 \quad , \quad 0\leq\theta\leq 2\pi$$

Also:
$$\begin{aligned}1-x^2-y^2 &= 1-(x^2+y^2) = 1-(r^2\cos^2\theta+r^2\sin^2\theta)\\ &=1-r^2(\sin^2\theta+\cos^2\theta)\\ &=1-r^2\end{aligned}$$

$$\Rightarrow V=\iint_D (1-x^2-y^2)\,dA=\int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$=\int_0^{2\pi} d\theta \int_0^1 (r-r^3)\,dr = 2\pi\cdot\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1$$

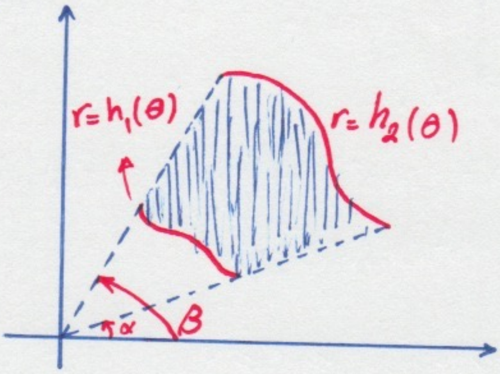
$$=2\pi\left[\frac{1}{2}-\frac{1}{4}\right)-\left[0-0\right]$$

$$\boxed{=2\pi\cdot\frac{1}{4}=\frac{\pi}{2}}$$

Note: This problem could be solved using rectangular coordinate system, however, the solution would be more complicated!

More Complicated Regions in Polar Coordinate:

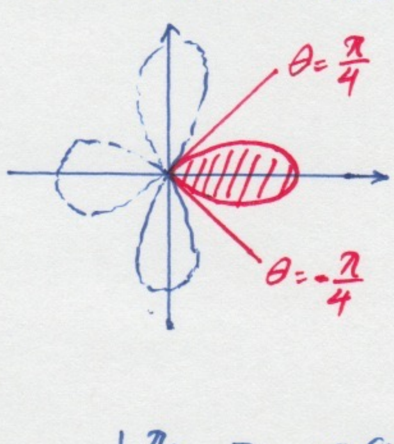
Sometimes, we have to deal w/ more complicated region such that r_1 and r_2 are NOT just numbers like $r_1 = a$ and $r_2 = b$.



Notice That r_1 and r_2 are now a relationship, $h_1(\theta)$ and $h_2(\theta)$ rather than Two numbers a and b. Now:

If f is continuous on polar region of the form:
$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$ Then $\displaystyle\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) \cdot r \cdot dr\,d\theta$

Example: use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$



Solution: $D = \{(r,\theta) \mid -\frac{\pi}{4} \le \theta \le \frac{\pi}{4},\ 0 \le r \le \cos 2\pi\}$

$$\Rightarrow A(\text{region } D) = \iint_D dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r \cdot dr \cdot d\theta$$

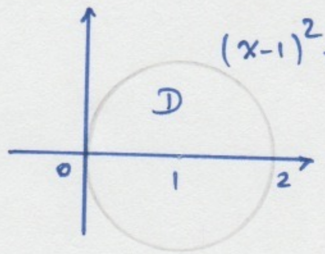
Remember, when we are dealing w/ area, $f(x,y) = 1$ so we do NOT write it.

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \left[\tfrac{1}{2} r^2\right]_0^{\cos 2\theta} d\theta = \tfrac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta = \tfrac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 4\theta)\,d\theta = \frac{\pi}{8}$$

Example: Find the Volume of the Solid that lies under the Paraboloid $z = x^2+y^2$, above the xy-plane and inside the cylinder $x^2+y^2 = 2x$.

Solution: $x^2+y^2 = 2x$ is the eq. of circle. After Completing the squares we have:

$(x-1)^2+y^2=1 \rightarrow$ if we graph this circle we will have

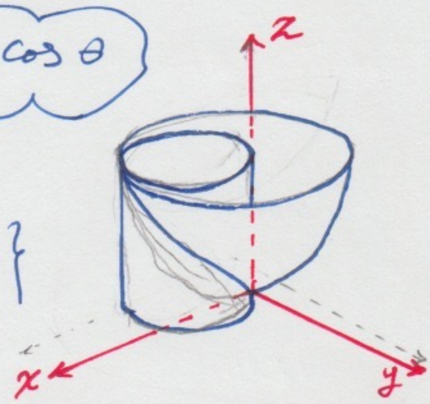


$(x-1)^2+y^2=1$, in polar coordinates we have

$$(r\cos\theta)^2+(r\sin\theta)^2 = 2\cdot r\cos\theta$$

$$\Rightarrow r^2 = 2r\cos\theta \Rightarrow \boxed{r = 2\cos\theta}$$

Also from the graph we see $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$



$$\Rightarrow \underset{\text{in polar}}{D} = \left\{(r,\theta) \,\middle|\, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2\cos\theta\right\}$$

From Paraboloid

$$V = \iint_D f(x,y)\,dA = \iint_D (x^2+y^2)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2\cdot r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos^4\theta}_{\text{even function}}\,d\theta = 8\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta = 8\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\Big[1+2\cos 2\theta + \overbrace{\tfrac{1}{2}(1+\cos 4\theta)}^{\cos^2 2\theta}\Big]\,d\theta$$

$$= 2\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\frac{\pi}{2}} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \boxed{\frac{3\pi}{2}}$$