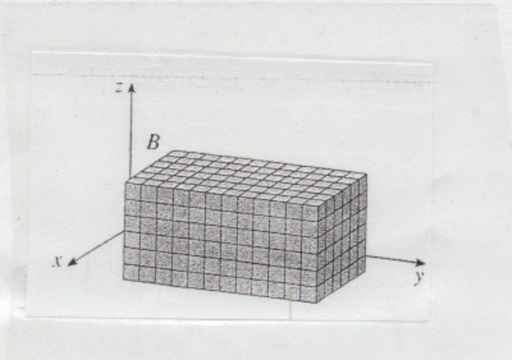
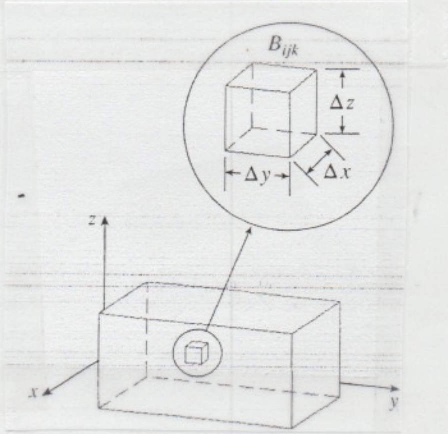
section 15.6

Similar to the way that we defined single integrals for functions of one Variable and double integrals for functions of two variables, we can define triple integrals for functions of three variables. The simplest case would be when f is defined on a rectangular Box:

$$B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$$





The first step is to divide B into Sub-Boxes. Then we see $\Delta V = \Delta x \Delta y \Delta y$

we can form triple Riemann Sum as:

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

* This is the Sample point picked inside a sub-box

By analogy with the definition of a double integral and a single integral, we define the triple integral as the limit of triple Riemann Sum:

$$\boxed{\iiint_B f(x,y,z)\,dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V}$$

Given that this limit exists.

we can get to a simpler-looking expression for the triple integral by picking the point (x_i, y_j, z_k) as the sample point:

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\,\Delta V$$

Fubini's Theorem for Triple Integrals:

Similar to what we learned for double integrals, the practical method for evaluating triple integrals is to use Fubini's Theorem (By expressing them as iterated integrals):

> If f is continuous on rectangular Box, $B = [a,b] \times [c,d] \times [r,s]$ then:
> $$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Note: This means that first we integrate w/ respect to x, then w/ respect to y and then w/ respect to z. Also, remember that we can change the order of integration w/ respect to any of these independent variables. The final answer should be the same.

Example: Evaluate the triple integral $\iiint_B xyz^2\,dV$, where B is the rectangular box given by:

$$B = \{(x,y,z) \mid 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}$$

Solution: we can pick any order for integration but let's stick to the $dx\,dy\,dz$ version:

$$\iiint_B xyz^2\,dV = \int_0^3\int_{-1}^2\int_0^1 xyz^2 \cdot dx\,dy\,dz = \int_0^3\int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_0^1 dy\,dz$$

$$= \int_0^3\int_{-1}^2 \frac{yz^2}{2}\,dy\,dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{-1}^2 dz = \int_0^3 \frac{3z^2}{4}\,dz = \frac{z^3}{4}\Big]_0^3 = \frac{27}{4}$$

Triple itegrals over a general bounded region E:

We define the triple integral over a general bounded region E in three dimensional space by similar procedure that we used for double integrals. We enclose E in box, B. Then, we define F so that it agrees w/ f on E but is <u>0</u> for the points inside box B and outside bounded region E.

In order to evaluate triple integrals, first we need to understand which type of region is E (The 3D region). We have Three different types of regions for E (The 3D region): Type I, Type II, Type III

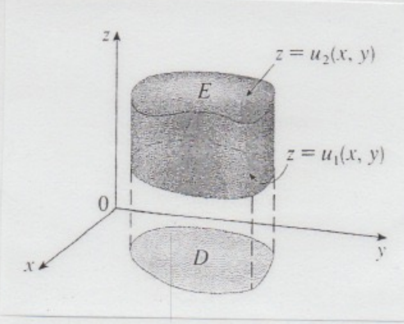
Hopefully, you still have <u>not</u> forgotten that the 2-Dimensional region has has two types: Two types of region D : Type I & Type II

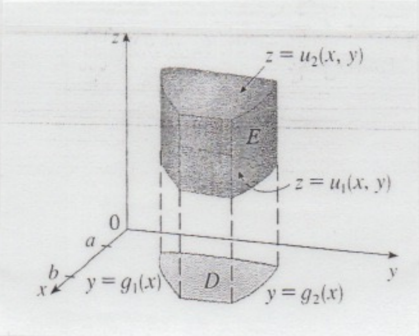
in the next page you will see why we care about both regions E and D.

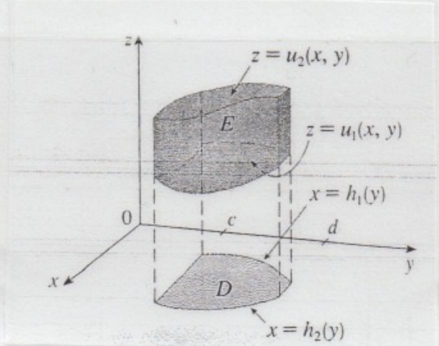
Type I of Region E: If the region lies between the graph of two continuous functions of $\underline{x}$ and $\underline{y}$, that is:

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

Roughly, you can think of it as the region E being elongate along z axis.







E: Type I
D: Type I

E: Type I
D: Type II

it can be shown that if E is a type I region:

$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$

Very Important: The projection of Region E onto xy-plane gives us region D which itself maybe type I or type II

1) In the case that region D is type I, we will have

$$E = \{(x,y,z) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x),\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

$$\Rightarrow \iiint_E f(x,y,z) = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \cdot dy \cdot dx$$

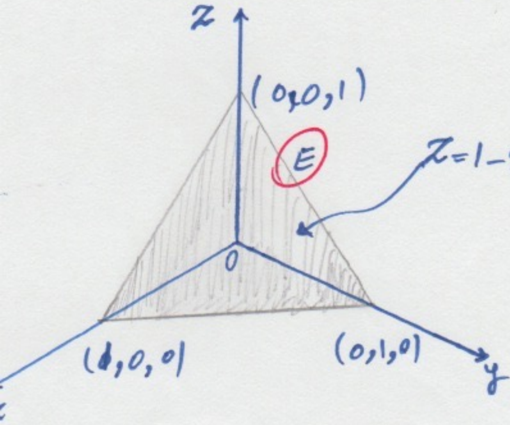
2) In the case that region D is a type II region, then:

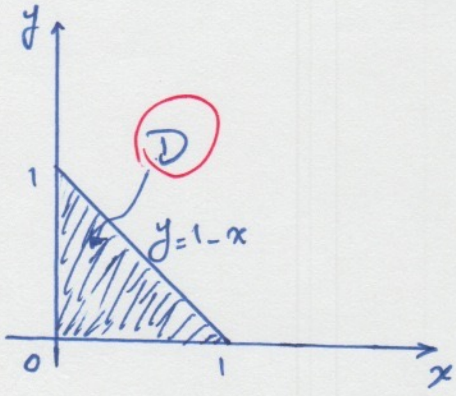
$$E = \{(x,y,z) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y),\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

$$\Rightarrow \iiint_E f(x,y,z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dx\, dy$$

Example: Evaluate $\iiint_E z\,dV$ where E is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$

Solution: Every time that we try to solve a problem involving a triple integral (setting up a triple integral), we should draw two diagrams: one that shows the solid region E, and one that shows its projection, D.





Notice that the lower boundary of the tetrahedron is the plane $z=0$ and the upper boundary is the plane $z=1-x-y \Rightarrow \begin{cases} u_1(x,y)=0 \\ u_2(x,y)=1-x-y \end{cases}$

Also, Notice that $z=0$ and $z=1-x-y$ intersect in the line $x+y=1$ in xy-Plane, which creates region D, Therefore:

$$\iiint_E z\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$= \int_0^1\int_0^{1-x}\frac{(1-x-y)^2}{2}\,dy\,dx = \frac{1}{2}\int_0^1\left[-\frac{(1-x-y)^3}{3}\right]_0^{1-x} dx$$

$$= -\frac{1}{6}\int_0^1\left[-\frac{(1-x)^3}{1}\right]dx = \frac{1}{6}\left[\frac{(1-x)^4}{4}\right]_0^1 = \frac{1}{24}$$

Type II of region E:

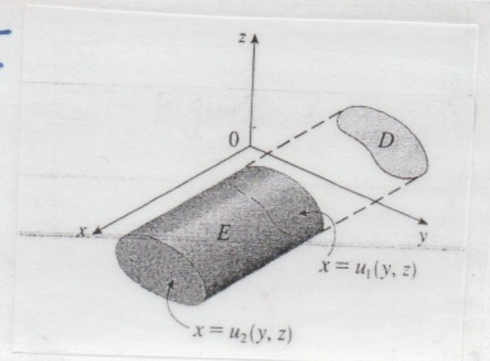
Similar to the discussion that we had in page 4, of the lecture notes, if region E is defined as: $E=\{(x,y,z) \mid (y,z)\in D,\ u_1(y,z)\leqslant x\leqslant u_2(y,z)\}$

This time D is the projection of E onto yz-plane (The back surface is $x=u_1(y,z)$, the front surface is $x=u_2(y,z)$ and we have:

$$\iiint_E f(x,y,z)\,dV=\iint_D\left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right]dA$$

Notice the difference

Remember, region D itself is a type I or type II



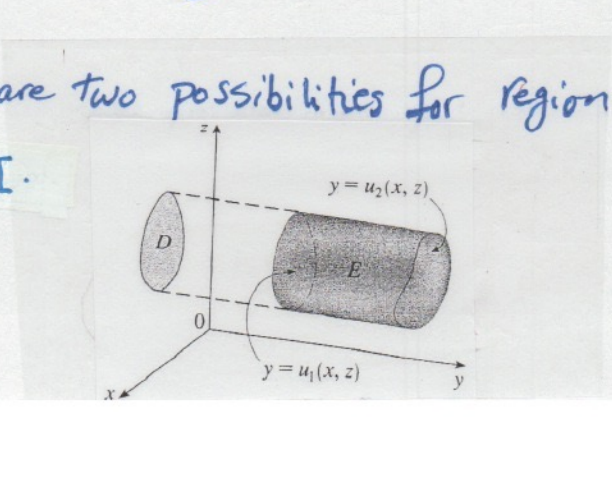
Type III of region E:

If region E is defined as:

$E=\{(x,y,z) \mid (x,z)\in D,\ u_1(x,z)\leqslant y\leqslant u_2(x,z)\}$ where D is the projection of E onto the xz-plane, $y=u_1(x,z)$ is the left surface, and $y=u_2(x,z)$ is the right surface. for this type of region:

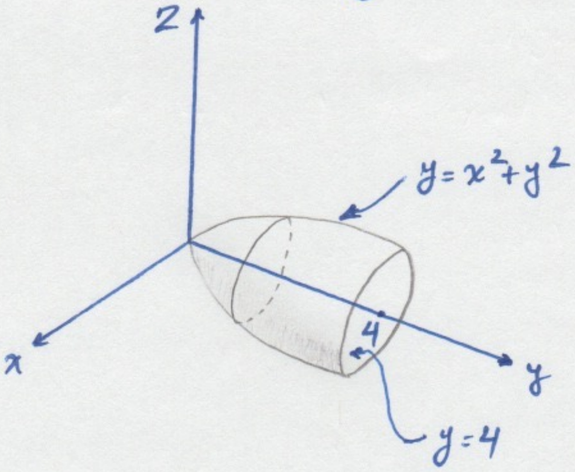
$$\iiint_E f(x,y,z)\,dV=\iint_D\left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right]dA$$

Again, in this case, there are two possibilities for region D, one is type I and the other is type II.



Example: Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$ where E is the region bounded by Paraboloid $y = x^2+z^2$ and the plane $y=4$.

Solution: If we go ahead and plot $y=x^2+z^2$ and $y=4$, we will have:



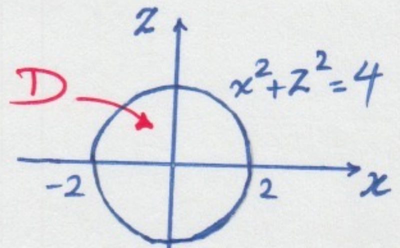
We can easily see that we may consider region E as type III region (elongated along y axis)

$$\Rightarrow \begin{cases} u_1(x,z) = x^2+z^2 \\ u_2(x,z) = 4 \end{cases}$$

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\right] dA$$

$$= \iint_D \left[y\sqrt{x^2+z^2}\right]_{y=x^2+z^2}^{y=4} dA = \iint_D \left[4-(x^2+z^2)\right]\sqrt{x^2+z^2}\, dA$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} \left(4-(x^2+z^2)\right)\sqrt{x^2+z^2}\, dz\, dx$$



It is much easier to convert this integral to Polar coordinates in order to solve it.

$$\Rightarrow \iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left(4-(x^2+z^2)\right)\sqrt{x^2+z^2}\, dA$$

$$= \int_0^{2\pi}\int_0^{2} (4-r^2)r.r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2 (4r^2-r^4)\,dr = 2\pi\left[\frac{4r^3}{3}-\frac{r^5}{5}\right] = \frac{128}{15}\pi$$

Example: Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^{y} f(x,y,z)\,dz\,dy\,dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

Solution: $$\int_0^1 \int_0^{x^2} \int_0^{y} f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV$$

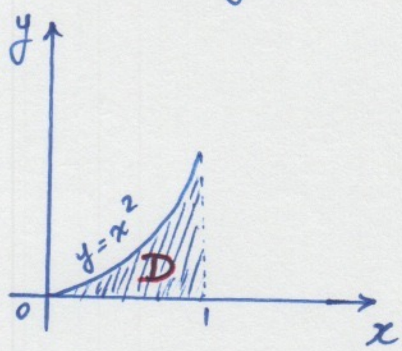
$$E = \{0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}$$

if we write projection of E onto xy-plane we will get:

$$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\}$$

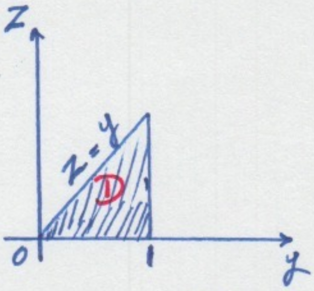
or

$$D = \{(x,y) \mid 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$$



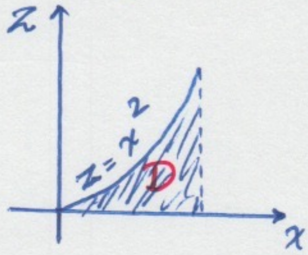
if we write projection of E onto yz-plane:

$$D = \{(y,z) \mid 0 \le y \le 1,\ 0 \le z \le y\}$$



if we write projection of E onto xz-plane:

$$D = \{(x,z) \mid 0 \le x \le 1,\ 0 \le z \le x^2\}$$



From the definition of region E we realize that it is the solid enclosed by $x=1$, $z=0$, $y=z$, and the parabolic cylinder $y=x^2$ (or $x=\sqrt{y}$). Since the order of integration starts from x, and the projection of E should be onto plane xy:

$$E = \{(x,y,z) \mid 0 \le y \le 1,\ 0 \le z \le y,\ \sqrt{y} \le x \le 1\}$$

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^{y} \int_{\sqrt{y}}^{1} f(x,y,z)\,dx\,dz\,dy$$