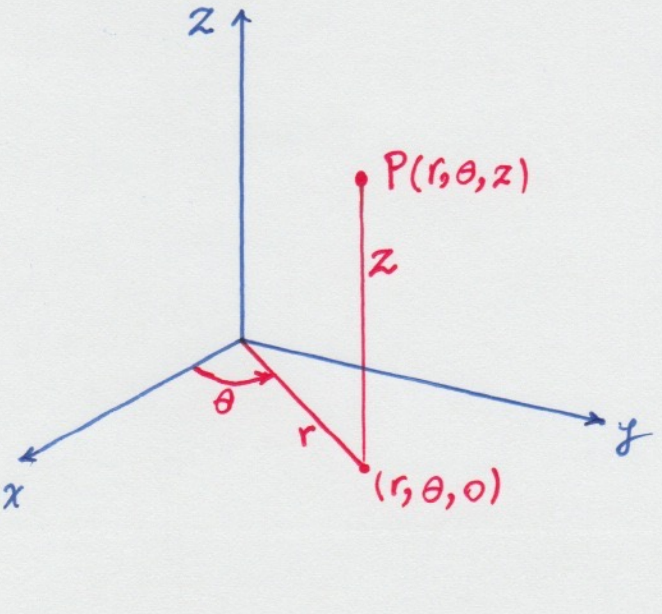
Introduction to Cylindrical Coordinates:

What was the reason that sometimes in plane geometry, we used polar coordinates instead of Cartesian Coordinates? The reason was that for some problems, Polar coordinates gave us a more convenient description of certain curves/Regions.

In three-dimensions (Not plane geometry or 2D), we define a new coordinate system called cylindrical coordinates, that is similar to Polar Coordinates except that it has an additional element.

Cylindrical Coordinates:

In The cylindrical Coordinate system, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r, and θ are polar coordinates of the projection of P onto xy-plane and z is the direct distance from xy-plane to P.



To convert from rectangular to cylindrical Coordinates, and vice versa, we use:

$$\boxed{\begin{array}{c} x = r\cos\theta \ , \ y = r\sin\theta \ , \ z = z \\ \hline r^2 = x^2 + y^2 \ , \ \tan\theta = \dfrac{y}{x} \ , \ z = z \end{array}}$$

Example: plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
Find cylindrical coordinates of the point w/ rectangular coordinates $(3, -3, -7)$.

Solution: we know $r=2$, $\theta = \frac{2\pi}{3}$, and $z=1$

$$\Rightarrow \left.\begin{array}{l} x = r\cos\theta = 2\cdot\cos\frac{2\pi}{3} = 2(-\tfrac{1}{2}) = -1 \\ y = r\sin\theta = 2\cdot\sin\frac{2\pi}{3} = 2(\frac{\sqrt{3}}{2}) = \sqrt{3} \\ z = 1 \end{array}\right\}$$

$(-1, \sqrt{3}, 1)$ is the coordinates of the point in rectangular coordinate system

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$x = 3$, $y = -3$, $z = -7$

$r^2 = x^2 + y^2 \Rightarrow r^2 = 3^2 + 3^2 = 18 \Rightarrow r = 3\sqrt{2}$

$\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{-3}{3} = -1 \Rightarrow \theta = \frac{7\pi}{4} + 2k\pi$

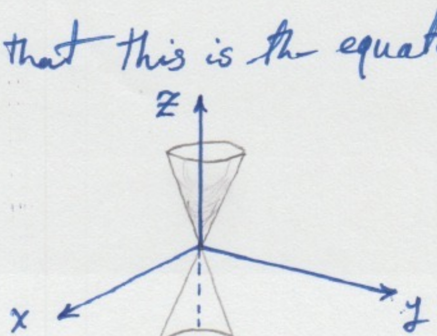
$z = -7 \Rightarrow (3\sqrt{2}, \frac{7\pi}{4}, -7)$ is the coordinates of the point in cylindrical coordinate system.

Note: cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry.

Example: Describe the surface whose equation in cylindrical coordinates is $z = r$.

$$z = r \Rightarrow \left.\begin{array}{l} z^2 = r^2 \\ r^2 = x^2 + y^2 \end{array}\right\} \Rightarrow z^2 = x^2 + y^2$$

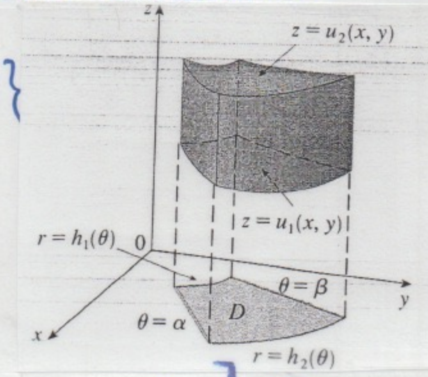
and we know that this is the equation of a circular cone whose axis is the z-axis



Evaluating Triple Integrals w/ Cylindrical Coordinates:

If E is a region (suppose that it is a type I region) whose projection D onto the xy-plane is conveniently described in polar coordinates:

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$



& D is given in polar coordinates by

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

we know that $\displaystyle\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$

and we know how to evaluate double integrals in polar coordinates, we can rewrite the above equation as:

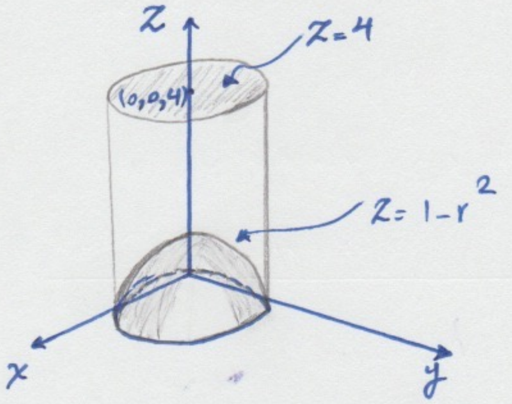
$$\boxed{\iiint_E f(x,y,z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta}$$

Note: The above equation says that to convert a triple integral from rectangular to cylindrical coordinates, we should write $x = r\cos\theta$, $y = r\sin\theta$, leaving z as it is, using appropriate limits of integration for z, r, and θ and replacing dV by $r\,dz\,dr\,d\theta$.

It is worthwhile to use cylindrical coordinates, especially when $f(x,y,z)$ involves x^2+y^2 expression.

Example: A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$ and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Solution: in cylindrical coordinates, the equation of a cylinder is $r=1 \to$ why?
The equation of a paraboloid is $z=1-r^2$ (why? Think about it)
if we draw region E, we will have:



$$E=\{(r,\theta,z)\mid 0\leq\theta\leq 2\pi,\ 0\leq r\leq 1,\ 1-r^2\leq z\leq 4$$

since the density at (x,y,z) is proportional to the distance from z-axis

$$\Rightarrow f(x,y,z) \propto \sqrt{x^2+y^2}$$

proportionality sign

$$\Rightarrow f(x,y,z) = k\sqrt{x^2+y^2}$$

proportionality Constant

$$\Rightarrow m=\iiint_E k\sqrt{x^2+y^2}\,dV=\int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (kr)\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 \left[kr^2\cdot z\right]_{1-r^2}^4 dr\,d\theta=\int_0^{2\pi}\int_0^1 kr^2\left(4-(1-r^2)\right)dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 \left(3kr^2+kr^4\right)dr\,d\theta=k\int_0^{2\pi}\left[r^3+\frac{r^5}{5}\right]_0^1 d\theta$$

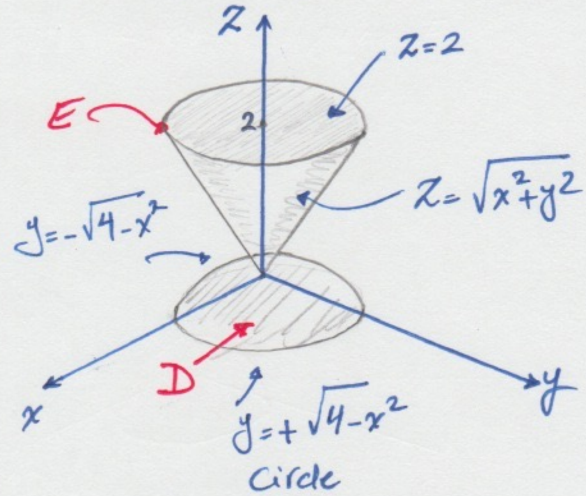
$$=k\int_0^{2\pi}\left(1+\frac{1}{5}\right)d\theta=k\left[\frac{6}{5}\theta\right]_0^{2\pi}=k\cdot\frac{6}{5}\cdot 2\pi=\frac{12k\pi}{5}$$

Example: Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz.\,dy.\,dx$.

solution: By looking at the integration bounds we see that

$$E=\left\{(x,y,z)\,\middle|\,-2\leq x\leq 2,\ -\sqrt{4-x^2}\leq y\leq +\sqrt{4-x^2},\ \sqrt{x^2+y^2}\leq z\leq 2\right\}$$

if we plot region E, we will have:



we see that the projection of E onto xy-plane is a circle w/ radius of 2.

The upper surface of E is plane $z=2$ and the lower surface of E is the cone $z=\sqrt{x^2+y^2}$

$$\Rightarrow E=\left\{(r,\theta,z)\,\middle|\,0\leq\theta\leq 2\pi,\ 0\leq r\leq 2,\ r\leq z\leq 2\right\}$$

$$\Rightarrow \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx=\iiint_E (x^2+y^2)\,dV$$

$$=\int_0^{2\pi}\int_0^2\int_r^2 r^2.\,r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^2 r^3\Big[z\Big]_r^2\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2 r^3(2-r)\,dr\,d\theta=\int_0^{2\pi}\left[\frac{r^4}{2}-\frac{r^5}{5}\right]_0^2 dr\,d\theta$$

$$=\int_0^{2\pi}\underbrace{\left(\frac{16}{2}-\frac{32}{5}\right)}\,d\theta=\int_0^{2\pi}\frac{8}{5}\,d\theta=\frac{8}{5}(2\pi-0)=\frac{16\pi}{5}$$