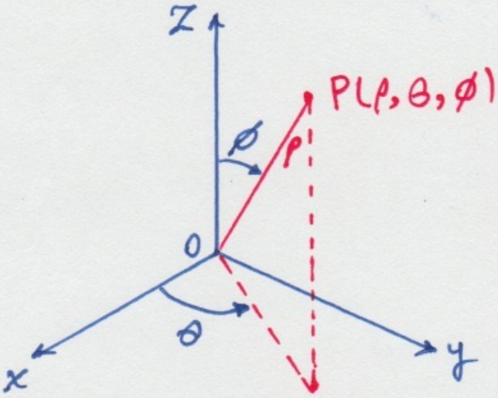
Section 15.8

Introduction:

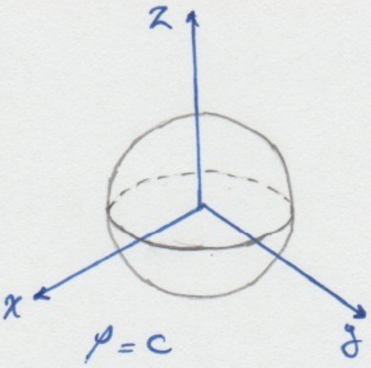
Spherical Coordinate system is another useful coordinate system in three dimensions that simplifies the evaluation of triple integrals over region bounded by spheres or cones.

We define the spherical coordinates of a point P in space as (ρ, θ, ϕ), where ρ is the distance from the origin to P, θ is the same as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and line-segment OP.



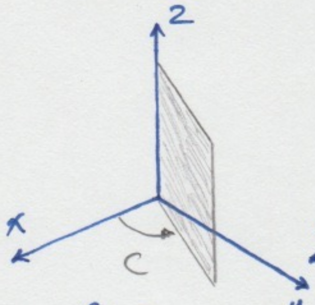
$\rho \geq 0$ and $0 \leq \phi \leq \pi$

Here is the list of four important surfaces that you need to know their equations in spherical coordinates:



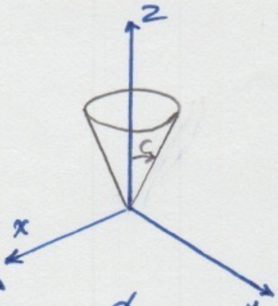
$\rho = c$

A sphere



$\theta = c$

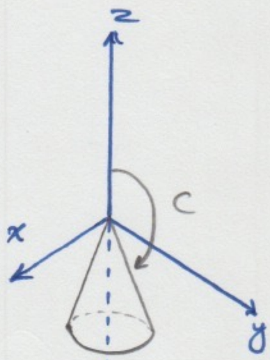
A half-plane



$\phi = c$

A half Cone

$0 < c < \frac{\pi}{2}$

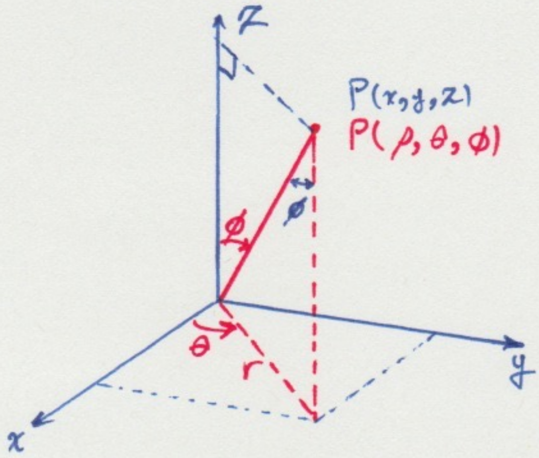


$\phi = c$

A half Cone

$\frac{\pi}{2} < c < \pi$

The relationship between rectangular and Spherical Coordinates:



$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

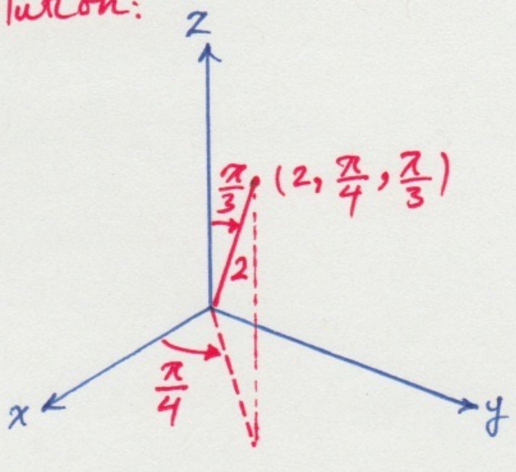
$z = \rho \cos\phi$

Also $\rho^2 = x^2 + y^2 + z^2$

Try to derive these 4 relationships using the provided graph.

Example: the point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ is given in spherical coordinates. Plot the point and find its rectangular Coordinates.

Solution:



we know that

$\rho = 2$, $\theta = \frac{\pi}{4}$, and $\phi = \frac{\pi}{3}$

$x = \rho \sin\phi \cdot \cos\theta = 2 \times \sin\frac{\pi}{3} \cdot \cos\frac{\pi}{4}$

$\Rightarrow x = 2 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$

$y = \rho \sin\phi \sin\theta = 2 \sin\frac{\pi}{3} \cdot \sin\frac{\pi}{4}$

$y = 2 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$

$z = \rho \cos\phi = 2 \times \frac{1}{2} = 1$

Therefore, point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ in spherical Coordinates is $(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1)$ in rectangular Coordinates.

Example: The point $(0, 2\sqrt{3}, -2)$ is given in rectangular Coordinates. Find the spherical Coordinates for this point.

Solution: $\rho^2 = x^2+y^2+z^2 \Rightarrow \rho^2 = 0^2 + 4\times 3 + 4 = 12+4 = 16 \Rightarrow \boxed{\rho = 4}$

$z = \rho\cos\phi \Rightarrow -2 = 4\cdot\cos\phi \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$

$x = \rho\sin\phi\cos\theta \Rightarrow 0 = 4\times\sin\frac{2\pi}{3}\cdot\cos\theta \Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \frac{\pi}{2}$

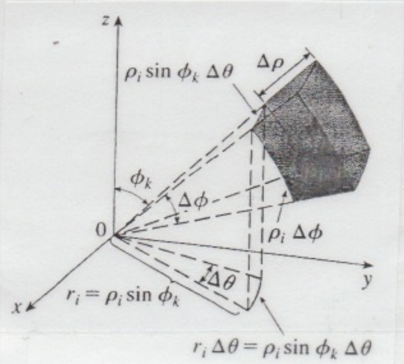
Note that θ can not be $\frac{3\pi}{2}$ because $y = 2\sqrt{3}$ is a positive number

$y = \rho\sin\phi\sin\theta \rightarrow \sin\frac{3\pi}{2} = -1$ and would make y A negative #.

Therefore, the spherical coordinates of the given point are $(4, \frac{\pi}{2}, \frac{2\pi}{3})$.

Evaluating Triple Integrals with Spherical Coordinates:

In spherical coordinate system, the counterpart of a rectangular box is a spherical wedge: $E = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$



it can be easily shown that the volume of spherical wedge can be calculated using similar procedure as the rectangular Box:

$$\Delta V_{ijk} = \rho_i^2 \sin\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

$$\iiint_E f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V_{ijk}$$

Therefore

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\,d\rho\,d\theta\,d\phi$$

where E is the spherical wedge given by:

$E = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$

remember that in order to convert points from rectangular coordinates to spherical coordinates we use:

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

The formula for triple integrals in spherical coordinate systems can be extended to regions such as:

$E = \{(\rho,\theta,\phi) \mid \alpha \le \theta \le \beta,\ c \le \phi \le d,\ g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)\}$

These will be the limit of integration for ρ

Note: usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Example: Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} \cdot dV$, where B is the unit ball:

$B = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$

Solution: we use spherical coordinates: $B = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$

we also know $x^2+y^2+z^2 = \rho^2$

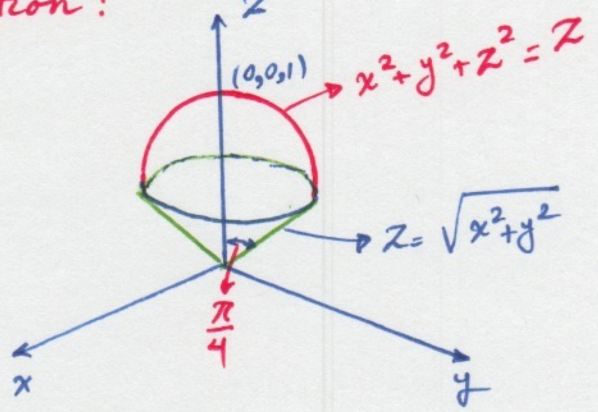
$$\Rightarrow \iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV = \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \cdot \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^1 \rho^2 e^{\rho^3} \cdot \sin\phi\,d\rho\,d\theta\,d\phi = \frac{4}{3}\pi(e-1)$$

Hint: to solve the inner integral assume $\rho^3 = u$

Example: use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$

solution:



The eq. of sphere $x^2+y^2+z^2=z$ can be written as:

$$\rho^2=\rho\cos\phi$$

The eq. of the cone $z=\sqrt{x^2+y^2}$ can be written as

$$\rho\cos\phi=\sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}$$

$$\rho\cos\phi=\sqrt{\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)}$$

$$\rho\cos\phi=\rho\sin\phi \Rightarrow \cos\phi=\sin\phi$$

$$\Rightarrow \phi=\frac{\pi}{4}$$

$$\Rightarrow E=\left\{(\rho,\theta,\phi)\mid 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{4},\ 0\le\rho\le\cos\phi\right\}$$

$$\Rightarrow V(E)=\iiint_E dV=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\frac{\rho^3}{3}\sin\phi\Big]_0^{\cos\phi}d\phi d\theta=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\frac{\cos^3\phi}{3}\cdot\sin\phi\cdot d\phi\, d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}-\frac{\cos^4\phi}{4}\Big]_0^{\frac{\pi}{4}}d\theta=\frac{-1}{3\times 4}\int_0^{2\pi}\left(\cos^4\frac{\pi}{4}-\cos^4 0\right)d\theta$$

$$=-\frac{1}{12}\int_0^{2\pi}\left(-\frac{3}{4}\right)d\theta=\frac{3}{4\times 12}(2\pi-0)=\frac{6\pi}{4\times 12}=\frac{\pi}{8}$$