In order to better understand the topic, we try to map this concept to change of variables (u-sub) that we learned in Math 2B (**Concept Mapping** is a very powerful method to understand new concepts by relating them to concepts that you have learned and mastered previously).

let's start by solving a single definite integral and going through all the details:

Ex. Evaluate $\int_0^2 x e^{x^2} \cdot dx$

1- First we substitute part of the integrand with a new variable.

$$u = x^2 \Rightarrow du = 2x \cdot dx \Rightarrow dx = \frac{du}{2x}$$

2- We evaluate the upper bound and lower bound for u.

$$\begin{cases} x_1 = 0 \\ x_2 = 2 \end{cases}, \quad u = x^2 \Rightarrow \begin{cases} u_1 = 0 \\ u_2 = 4 \end{cases}$$

3- we replace all the x's with a function of u, simplify the integrand and solve the integral.

$$\int_0^2 x e^{x^2} \cdot dx = \int_0^4 x \cdot e^u \cdot \frac{du}{2x} = \frac{1}{2}\int_0^4 e^u du = \frac{1}{2}\left(e^4 - 1\right)$$

* Now, let's apply exactly the same procedure for a general integral

$\int_a^b f(x)\,dx$

1- $u = g(x) \Rightarrow du = g'(x)\,dx = \frac{dg}{dx} \cdot dx = \frac{du}{dx} \cdot dx$

if we multiply both sides by $\frac{dx}{du}$, $du \cdot \frac{dx}{du} = \frac{du}{dx} \cdot \frac{dx}{du} \cdot dx$

$$\Rightarrow dx = \frac{dx}{du} \cdot du$$

You could also think about it this way: since

$u = g(x) \Rightarrow x = h(u)$, $dx = h'(u)\,du = \frac{dh}{du} \cdot du$

$x = h(u) \Rightarrow dx = \frac{dx}{du} \cdot du$

2- we evaluate the upper bound and lower bound for u.

$$\begin{cases} x_1 = a \\ x_2 = b \end{cases} \Rightarrow \begin{cases} u_1 = c \\ u_2 = d \end{cases}$$

3- Since we picked $u = g(x)$, this means $x = h(u)$. now, we replace all the x's with $h(u)$ in the integral and we simplify.

Therefore: $$\int_a^b f(x)\,dx = \int_c^d f(h(u)) \cdot \frac{dx}{du} \cdot du$$

This is the same as
$h'(u)\,du$

I rewrote this section and this is why half the page is blank very similar to the change of variable for a single integral, we have a change of variable for a double integral. we have two variables for a double integral (x, and y). So, we need to have two new variables for substitution, u, and v.

$x = h(u)$ for single integrals

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$ for double integrals.

instead of $\frac{dx}{du}$ we have something new called Jacobian.

$$\int_a^b f(x)\,dx = \int_c^d f(h(u))\,\frac{dx}{du}\cdot du$$

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v),\,h(u,v))\cdot J\cdot du\,dv$$

The region gets converted too

in single integrals we change the variable to be able to deal w/ integrand but in double integrals we typically apply change of variable to deal with region and convert a nasty region to a workable region. (convert parallelogram to rectangle or to convert an elliptical region to a circular region)

Jacobian:

Now, let's see what Jacobian is (J in the formula)

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

Example: Suppose that we have the following transformations:
$x = 2u + v$ and $y = u^2 - v$. find the Jacobian of This transformation.

Solution:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 2u & -1 \end{vmatrix} = -2 - 2u = -2(u+1)$$

Example: Find the Jacobian for the following transformation:

$$x = 2u + w \qquad y = u^2 - v^2 \qquad z = u + v^2 - 2w^2$$

solution:

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 2u & -2v & 0 \\ 1 & 2v & -4w \end{vmatrix}$$

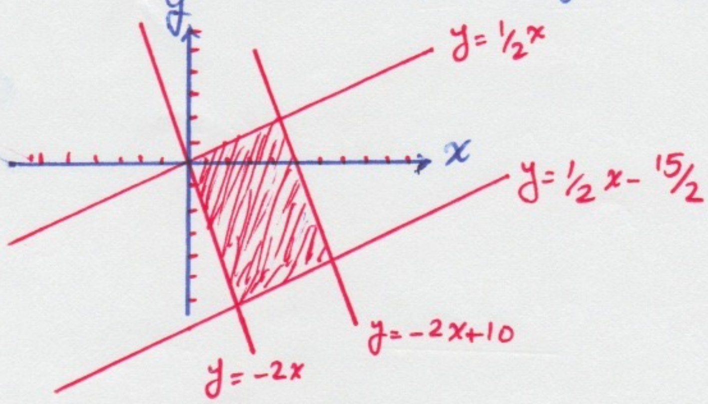
$$J = 2\begin{vmatrix} -2v & 0 \\ 2v & -4w \end{vmatrix} - 0\begin{vmatrix} 2u & 0 \\ 1 & -4w \end{vmatrix} + 1\begin{vmatrix} 2u & -2v \\ 1 & 2v \end{vmatrix}$$

$$J = 2(8vw - 0) - 0 + 4uv + 2v$$

$$\Longrightarrow \boxed{J = 16vw + 4uv + 2v}$$

Example: Evaluate $\iint_R (x+y)\,dA$ where R is the region bounded by $y=-2x$, $y=\frac{1}{2}x-\frac{15}{2}$, $y=-2x+10$, $y=\frac{1}{2}x$. Suppose that the following transformation is given: $\begin{cases} x = u+2v \\ y = v-2u \end{cases}$

solution: let's draw the region first:



As we can see, region R is not an easy to work region. it's also not a good candidate for polar coordinate systems. So, the next option is change of variable.

$$\iint_R f(x,y)\,dA = \iint_R (x+y)\,dA \xrightarrow{\text{Convert}} \iint_S f(g(u,v), h(u,v))\, J \cdot du\,dv$$

Now, we go ahead and plug-in the transformations that we have in the equation of boundaries:

$y=-2x \Rightarrow v-2u=-2(u+2v) \Rightarrow v-\cancel{2u}=-\cancel{2u}-4v \Rightarrow 5v=0$

$\Rightarrow \boxed{v=0}$

$y=\frac{1}{2}x-\frac{15}{2} \Rightarrow 2y=x-15$, $2(v-2u)=u+2v-15 \Rightarrow 5u=15$

$\Rightarrow \boxed{u=3}$

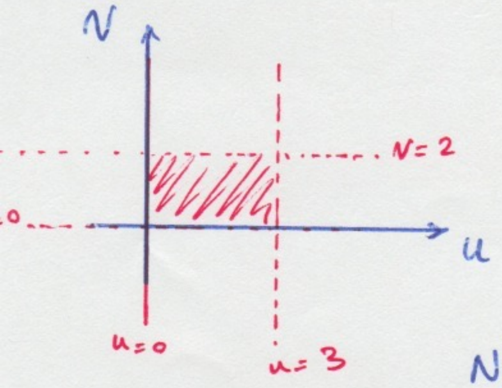
$y=-2x+10 \Rightarrow v-2u=-2(u+2v)+10 \Rightarrow v-2u=-2u-4v+10$

$5v=10 \Rightarrow \boxed{v=2}$

$y=\frac{1}{2}x \Rightarrow 2y=x \Rightarrow 2(v-2u)=u+2v \Rightarrow -3u=0$

$\Rightarrow \boxed{u=0}$

Now, $u=0$, $v=2$, $u=3$, $v=0$ are the boundaries in the new region that we have defined in uv-axis.



$$\Rightarrow \iint_R (x+y)\,dA = \int_0^3 \int_0^2 (\underbrace{u+2v}_{x} + \underbrace{v-2u}_{y})\, J \,dv\,du$$

$$= \int_0^3 \int_0^2 (3v-u)\, J \,dv\,du$$

Now, the only remaining part is to calculate Jacobian.

$x = u+2v$
$y = v-2u$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\Rightarrow J = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1-(2\times(-2)) = 5 \Rightarrow J = 5$$

$$\Rightarrow \iint_R (x+y)\,dA = \int_0^3 \int_0^2 (3v-u)\,5\,dv\,du = 5\int_0^3 \frac{3}{2}v^2 - uv\Big]_0^2 du$$

$$= 5\int_0^3 \left[\left(\frac{3}{2}\times 4 - 2u\right) - (0-0)\right] du = 5\int_0^3 (6-2u)\,du = 6u-u^2\Big]_0^3 \times 5$$

$$= 5\,(6\times 3 - 9) = 45$$

Example: Evaluate the following integral:

$$\iint_R \sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\, dA \text{ where region } R \text{ is an ellipse w/ eq. } \frac{x^2}{4}+\frac{y^2}{9}=1.$$

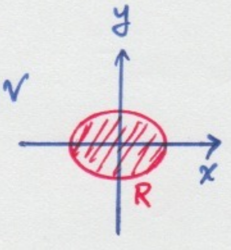
Solution: the first thing that we notice about this problem is that the transformation is not given to us. so, how can we find this transformation? Also notice that region R is the equation of an ellipse so we need to find a transformation that shrinks region R (ellipse) to region S (circle) so we can use polar coords to solve this problem.

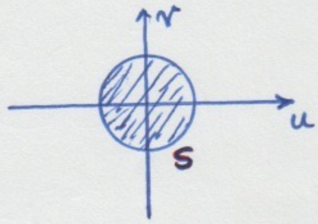
if we pick this transformation $\begin{cases} x=2u \\ y=3v \end{cases}$, we see that $\frac{(2u)^2}{4}+\frac{(3v)^2}{9}=1$

$\Rightarrow u^2+v^2=1$

$$\iint_R \sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\, dA = \iint_S \sqrt{1-u^2-v^2}\cdot J\, du\, dv$$

Notice the difference





$$J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 \Rightarrow \text{we have } \iint_S \sqrt{1-(u^2+v^2)}\cdot \underset{\text{Jacobian}}{6}\cdot du\, dv$$

Now we can convert this new integral to a double integral in polar coordinate system. $\begin{cases} u=r\cos\theta \\ v=r\sin\theta \end{cases}$

$$\Rightarrow 6\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\cdot r\cdot dr\, d\theta = 6\int_0^{2\pi} \left(-\tfrac{1}{3}\right)\left(\sqrt{1-r^2}\right)^3\Big]_0^1 d\theta$$

$$= -2\int_0^{2\pi} (0-1)\, d\theta = 2\int_0^{2\pi} d\theta = 4\pi$$

Example: prove that if we have a transformation like $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian of this transformation is r.

solution: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$$\Rightarrow J = (r\cos\theta)\cos\theta - (-r\sin\theta)\sin\theta = r\cos^2\theta + r\sin^2\theta$$
$$= r$$

$$\Rightarrow \boxed{J = r}$$

Therefore, in polar coordinate transformation, when we convert $dxdy$ to $r dr d\theta$ we really convert $dxdy$ to $J dr d\theta$, like all other transformations.

Example: Find $\iint_R xy^2 dA$, where R is the region bounded by $xy=1$, $xy=2$, $y=x$ and $y=2x$.

Solution: Again, we do not have a transformation in this problem but if we look at $xy=1$ and $xy=2$, the first thing that comes to mind is that we need to pick u and v such that when we plug them in eqs. (e.g. $xy=1$), one variable gets cancelled out and we get $u=$const or $v=$Cons.

$$\begin{cases} x = \frac{u}{v} \\ y = v \end{cases} \Rightarrow xy = 1 \Rightarrow \frac{u}{v}\cdot v = 1 \Rightarrow u = 1$$
$$xy = 2 \Rightarrow \frac{u}{v}\cdot v = 2 \Rightarrow u = 2$$
$$y = x \Rightarrow \frac{u}{v} = v \Rightarrow v^2 = u$$
$$y = 2x \Rightarrow v = 2\frac{u}{v} \Rightarrow v^2 = 2u$$

$$\Rightarrow \iint_R xy^2 dA = \int_{u=1}^{u=2} \int_{v=\sqrt{u}}^{v=\sqrt{2u}} \frac{u}{v}\cdot v^2 \cdot J \, dv\, du$$

$$\Rightarrow \quad = \int_1^2 \int_{\sqrt{u}}^{\sqrt{2u}} uv \cdot J \cdot dv\, du$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/v & -\frac{u}{v} \\ 0 & 1 \end{vmatrix} = 1/v$$

$$= \int_1^2 \int_{\sqrt{u}}^{\sqrt{2u}} \cdot u \cdot dv\, du$$

$$= \int_1^2 uv\Big]_{v=\sqrt{u}}^{v=\sqrt{2u}} du$$

$$= \int_1^2 \left(u\sqrt{2u} - u\sqrt{u}\right) du = (\sqrt{2}-1)\int_1^2 u^{3/2} \cdot du = (\sqrt{2}-1)\frac{2}{5} \cdot u^{5/2}\Big]_1^2$$

$$= (\sqrt{2}-1)\frac{2}{5} \cdot \left(2^{5/2} - 2^1\right) = \frac{2}{5}\left(9 - 5\sqrt{2}\right)$$

How Can we find the transformation:

In general, if you are not given the transformation, start by picking the tricky part of the integrand or the region given:

Example: $\iint_R e^{\frac{x+y}{x-y}}\, dA \xrightarrow{\text{we Can Pick}} \begin{cases} u = x+y \\ v = x-y \end{cases}$

Example: $R: \begin{cases} x+y = -1 \\ x+y = 3 \\ 2x-y = 0 \\ 2x-y = 4 \end{cases} \longrightarrow \begin{matrix} u = x+y \\ v = 2x-y \end{matrix} \longrightarrow \begin{cases} -1 \le u \le 3 \\ 0 \le v \le 4 \end{cases}$

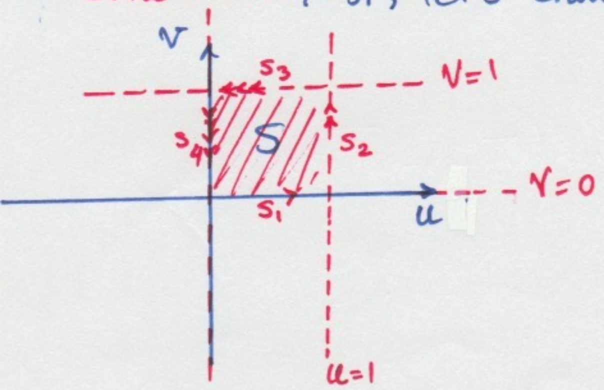
practice example: prove that in spherical coordinate system, Jacobian of the transformation is $J = \rho^2 \sin\phi$.

Example: A transformation is defined by the equations:

$$x = u^2 - v^2$$
$$y = 2uv$$

Find the image of the square $S = \{(u,v) \mid 0 \le u \le 1, 0 \le v \le 1\}$

Solution: first, let's draw region S in uv-axis:



S_1: $v=0$, $0 \le u \le 1 \Rightarrow x = u^2 \Rightarrow 0 \le x \le 1$, $y=0$

S_2: $u=1$, $0 \le v \le 1 \Rightarrow x = 1 - v^2$, $y = 2v$

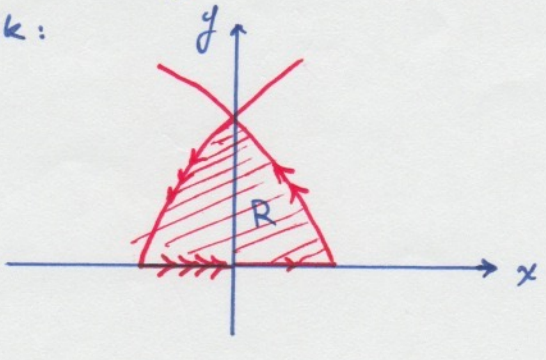
$0 \le y \le 2 \Rightarrow x = 1 - \frac{y^2}{4}$ $0 \le x \le 1$

S_3: $v=1$, $0 \le u \le 1 \Rightarrow x = u^2 - 1$, $y = 2u$

$0 \le y \le 2 \Rightarrow x = \frac{y^2}{4} - 1$ $-1 \le x \le 0$

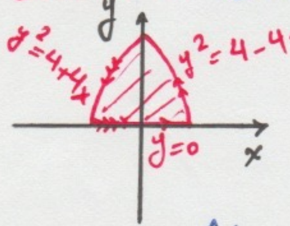
S_4: $u=0$, $0 \le v \le 1 \Rightarrow -1 \le x \le 0$, $y=0$

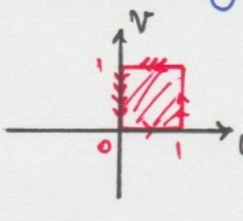
Now, if we go ahead and plot this region in xy-plane, how it would look:



Example: use the change of variable $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4-4x$ and $y^2 = 4+4x$, $y \geq 0$.

Solution: let's calculate Jacobian first: $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$





(Similar to the previous problem)

$\Rightarrow J = 4u^2 + 4v^2$

$$\Rightarrow \iint_R y\,dA = \iint_S 2uv \cdot 4(u^2+v^2)\,du\,dv = 4\int_0^1\int_0^1 2uv\,(u^2+v^2)\,du\,dv$$

$$= 8\int_0^1\int_0^1 (u^3v + uv^3)\,du\,dv = \int_0^1 (2v+4v^3)\,dv = 2$$

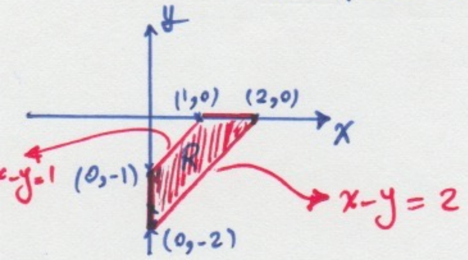
Note: finding the definition of region S in terms of u, and v was only possible by using the previous problem's results, otherwise, it would by very hard to do so!

Example: Evaluate the integral:

$\iint_R e^{x+y/x-y}\,dA$ where R is the region between the vertices (1,0), (2,0) (0,−2), and (0,−1).

$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

Solution: we pick $u = x+y$, $v = x-y \Rightarrow \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$



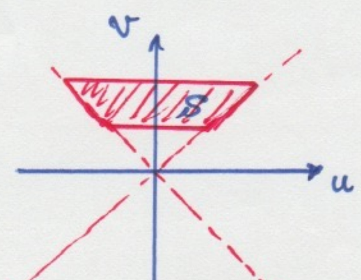
the sides of region R in xy-plane are

$y=0$, $x-y=2$, $x=0$, $x-y=1$

plug in the transformations to get the boundaries of region S as following

$u=v$, $v=2$, $u=-v$, $v=1$

$\Rightarrow S = \{(u,v) \mid 1 \leq v \leq 2,\ -v \leq u \leq v\}$



$$\iint_R e^{\frac{x+y}{x-y}}\,dA = \int_1^2\int_{-v}^{v} e^{u/v}\left(\tfrac{1}{2}\right)du\,dv = \tfrac{1}{2}\int_1^2 v e^{u/v}\Big]_{-v}^{v}\,dv$$

$$= \tfrac{1}{2}\int_1^2 (e-e^{-1})\,v\,dv = \tfrac{3}{4}(e-e^{-1})$$