Section 16.1

Introduction to vector fields:

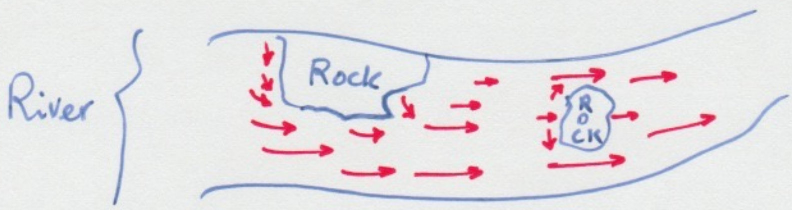
So far, every function that we have dealth with takes an input (A scalar input in the form of x, (x,y), or (x,y,z)) and gives a scalar output from the other side

$$x \longrightarrow \boxed{f} \longrightarrow f(x)$$

$$(x,y) \longrightarrow \boxed{f} \longrightarrow f(x,y)$$

$$(x,y,z) \longrightarrow \boxed{f} \longrightarrow f(x,y,z)$$

in chapter 16, we will deal with a different type of function and that is a vector field or a vector function. A vector field is a function That assigns a vector to any point in the field. Think of a river that at any Point of it, the velocity vector might be different in terms of direction and magnitude.



At any point inside this river we Can assign a vector that has its own direction and magnitude. This is what we call a vector field.

Definition: Let D be a set in $\mathbb{R}^2$ (a plane region). A vector field on $\mathbb{R}^2$ is a function F that assigns to each point (x,y) in D a two dimensional vector $\vec{F}(x,y)$. $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} = \langle P(x,y), Q(x,y) \rangle$

Definition: let E be a set in $\mathbb{R}^3$ (in space). A vector field on R^3 is a function F that assigns to each point (x,y,z) in E a three-dimensional vector $\vec{F}(x,y,z)$. $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$

Example: A vector field on $\mathbb{R}^2$ is defined by $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$. Describe F by sketching some of the vectors $\vec{F}(x,y)$.

Solution: we know that a vector field is a function that assigns a vector to each point in the field. Therefore we pick several different points and then we try to find the vector associated w/ that point. Then we sketch all these vectors.

Note: the starting point of these vectors are always the (x,y) value of the point that we pick.

(x,y)

$(0,1) \longrightarrow F(x,y) = -\vec{i}$

$(0,2) \longrightarrow F(x,y) = -2\vec{i}$

$(0,-1) \longrightarrow F(x,y) = \vec{i}$

$(0,-2) \longrightarrow F(x,y) = 2\vec{i}$

$(1,0) \longrightarrow F(x,y) = \vec{j}$

$(2,0) \longrightarrow F(x,y) = 2\vec{j}$

$(-1,0) \longrightarrow F(x,y) = -\vec{j}$

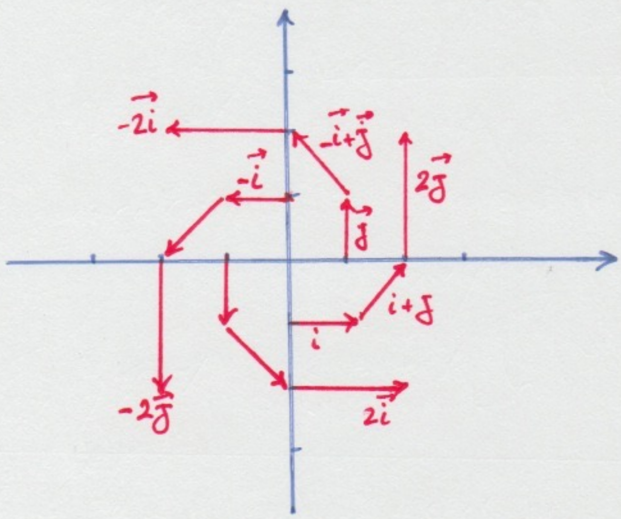
$(-2,0) \longrightarrow F(x,y) = -2\vec{j}$

$(1,1) \longrightarrow F(x,y) = -\vec{i} + \vec{j}$

$(-1,1) \longrightarrow F(x,y) = -\vec{i} - \vec{j}$

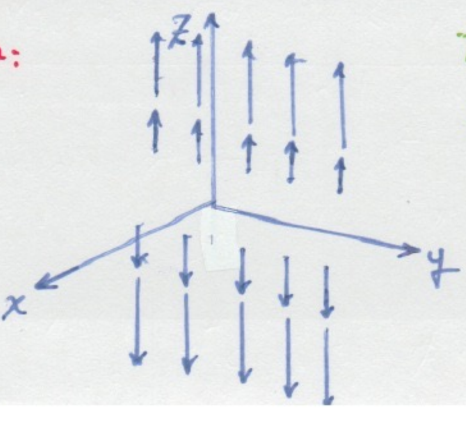
$(1,-1) \longrightarrow F(x,y) = \vec{i} + \vec{j}$

$(-1,-1) \longrightarrow F(x,y) = \vec{i} - \vec{j}$



Example: sketch the vector field on $\mathbb{R}^3$ given by $F(x,y,z) = z\vec{k}$

Solution:



These vectors are all parallel to z axis.

Example: sketch the vector field on $\mathbb{R}^2$ given by $\vec{F}(x,y) = x\vec{i} + y\vec{j}$.

Solution: if we pick several different points and sketch the vectors associated w/ the points, we will have

$(0,1) \Rightarrow \vec{F}(x,y) = \vec{j}$

$(1,1) \Rightarrow \vec{F}(x,y) = \vec{i} + \vec{j}$

$(1,0) \Rightarrow \vec{F}(x,y) = \vec{i}$

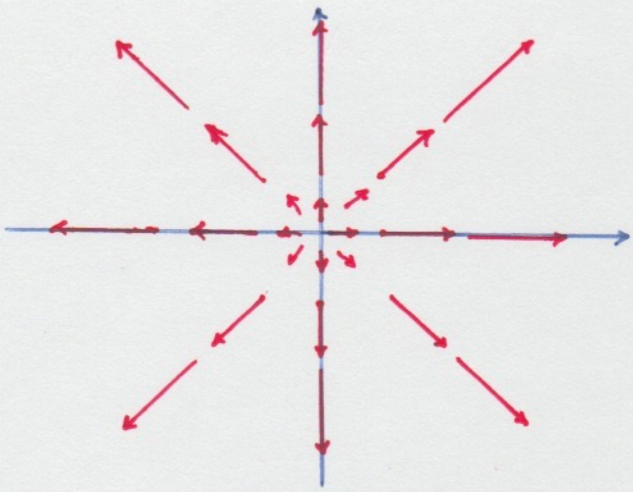
$(-1,0) \Rightarrow \vec{F}(x,y) = -\vec{i}$

$(0,-1) \Rightarrow \vec{F}(x,y) = -\vec{j}$

$(1,-1) \Rightarrow \vec{F}(x,y) = \vec{i} - \vec{j}$

$(-1,1) \Rightarrow \vec{F}(x,y) = -\vec{i} + \vec{j}$

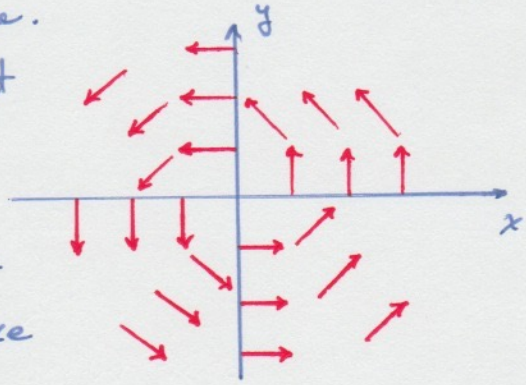
$(-1,-1) \Rightarrow \vec{F}(x,y) = -\vec{i} - \vec{j}$



Example: sketch the vector field on $\mathbb{R}^2$ given by $\vec{F}(x,y) = \dfrac{-\vec{i}y + x\vec{j}}{\sqrt{x^2+y^2}}$

Solution: This is an interesting field. if you notice $\sqrt{x^2+y^2}$ is the magnitude of the position vector at all (x,y) points. Therefore, the magnitude of all these vectors are the same.

if you pick different points in xy-plane and find the vectors you will see that the vector field looks like this.



Gradient Fields:

From Math 2D you remember that if f is a scalar function of two variables, ∇f (grad f) is defined by $\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$

Therefore, ∇f is really a vector field on $\mathbb{R}^2$ and is called gradient vector field.

we can define $\nabla f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}$ in $\mathbb{R}^3$ as The gradient vector field on $\mathbb{R}^3$.

Example: Find the gradient vector field of $f(x,y) = x^2y - y^3$.

Solution: $f(x,y)$ is a scalar function and $\begin{cases} f_x = \dfrac{\partial f}{\partial x} = 2xy \\ f_y = \dfrac{\partial f}{\partial y} = x^2 - 3y^2 \end{cases}$

$$\Rightarrow \nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = (2xy)\vec{i} + (x^2 - 3y^2)\vec{j}$$

Conservative vector field and potential function:

A vector field F is called a conservative vector field if it is the gradient of some scalar function, that is, if there exists a function f such that $F = \nabla f$. In this situation f is called a potential function for F.

Note: Not all vector fields are conservative, but such fields do arise frequently in physics.

e.g. gravitational field $\vec{F}$ around earth is conservative because we define the potential function $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$, Then $\nabla f(x,y,z) = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j} + \dfrac{\partial f}{\partial z}\vec{k}$

and $\nabla f(x,y,z) = F(x,y,z) \rightarrow$ Gravitational field.