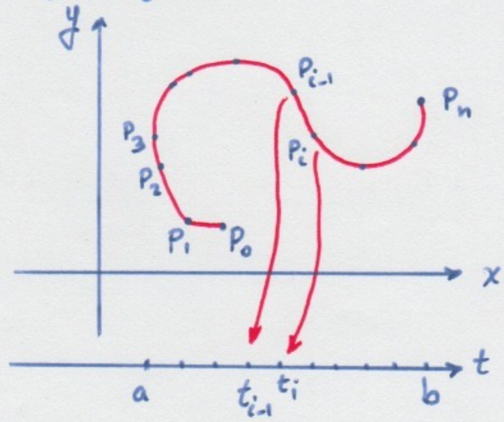
Introduction to line integrals:

line integrals are similar to single integrals except that instead of integrating over an interval $[a,b]$, we integrate over a curve (A path).



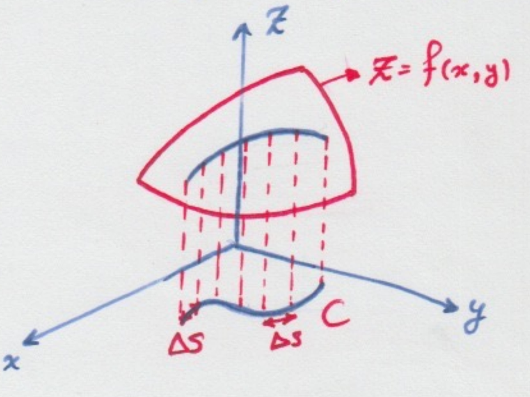
from math 2D, you remember That The best way to deal with such curves is by parametric equation.

$$x = x(t)\ ,\ y = y(t) \quad a \leq t \leq b$$

or equivalently by vector eq.

$$r(t) = x(t)\vec{i} + y(t)\vec{j}$$

Similar to the idea of double integrals that $\iint_D f(x,y)\,dA$ meant area of small pieces of region D multiplied by the height (The distance from (x_i^*, y_i^*) to the surface $f(x,y)$), line integrals convey a similar message. Except that instead of region D, we have a curve C.



$$\Rightarrow \int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta S_i$$

Definition of line integrals:

If f is defined on a smooth curve C given by parametric eq. $x = x(t)$ and $y = y(t)$ for $a \leq t \leq b$, then the line integral of f along C is:

$$\int_C f(x,y)\,dS = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta S_i$$

we know that $l = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ and $dS = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,.dt$

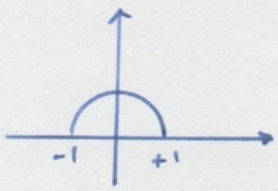
$$\Rightarrow \int_C f(x,y)\,dS = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

↳ when we write the line integral in terms of parametric eq.

Note: By looking at the graph in the previous page you can see that we can interpret the line integral of a positive function as an area. In fact, if $f(x,y) \geqslant 0$, $\int_C f(x,y)\,ds$ represents the area of one side of the fence or the curtain created b/w the curve C in xy-plane and its projection on $f(x,y)$ in space.

Example: Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of the unit circle $x^2+y^2=1$.

Solution: The first thing that we do, is to convert the curve C to its parametric equivalent.



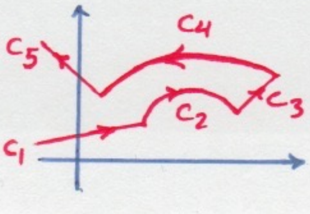
$$\begin{cases} x = r\cos t \\ y = r\sin t \\ r = 1 \end{cases} \Rightarrow \begin{array}{l} x(t) = \cos t \\ y(t) = \sin t \end{array}$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$\Rightarrow \int_C (2+x^2y)\,ds = \int_0^\pi (2+\cos^2 t\sin t)\sqrt{\cos^2 t+(-\sin t)^2}\,.\,dt$$

$$= \int_0^\pi (\sin t\cos^2 t+2)\,dt = \left[2t-\frac{\cos^3 t}{3}\right]_0^\pi = 2\pi+\frac{2}{3}$$

Note: Sometimes the integration path (C) is not a single curve but it is a piecewise-smooth curve, which means, it is a union of a finite number of curves $C_1, C_2, \dots, C_n$. In these cases we define the line integral along C as the sum of the integrals of $f(x,y)$ along each of the smooth pieces of C:

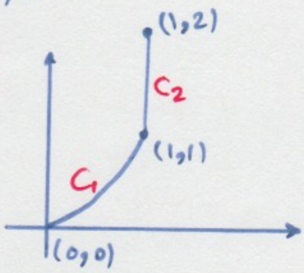


$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \dots + \int_{C_n} f(x,y)\,ds$$

Example: Evaluate $\int_C 2x\, ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

(The solution of this problem is slightly different than your book's)

solution:



$x = t$ (we assume) $\Rightarrow y = t^2$

remember we have to make sure that the interval that we choose for t satisfies the initial and the end point of the curve. if it does not, then we change the t interval. (you will see problem like that in this section)

Part 1:

$$\begin{cases} t = 0 \rightarrow (0,0) \text{ initial point} \\ t = 1 \rightarrow (1,1) \text{ end point} \end{cases} \Rightarrow 0 \leq t \leq 1$$

All we are saying here is that we can pick t to be between 0 and 1 just because they satisfy the points on the parametric eq.

$$\Rightarrow \int_{C_2} f(x,y)\, ds = \int_0^1 2(t)\sqrt{(1)^2 + (2t)^2}\, \cdot dt$$

$$= \int_0^1 2t\sqrt{1+4t^2}\, dt = \frac{5\sqrt{5}-1}{6}$$

Assume that this is u and solve the integral

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Part 2: The parametric eq. for a line segment looks like $\begin{cases} x = c_1 + k_1 t \\ y = c_2 + k_2 t \end{cases}$

for line segments we always pick $0 \leq t \leq 1$ as that gives us the easiest way to find the parametric eq. of the line segment

$$\left.\begin{aligned} &\Rightarrow x(t) = c_1 + k_1 t \xrightarrow[(1,1)]{t=0} 1 = c_1 + k_1(0) \Rightarrow c_1 = 1 \\ &y(t) = c_2 + k_2 t \xrightarrow[(1,1)]{t=0} 1 = c_2 + k_2(0) \Rightarrow c_2 = 1 \\ &x(t) = 1 + k_1 t \xrightarrow[(1,2)]{t=1} 1 = 1 + k_1(1) \Rightarrow k_1 = 0 \\ &y(t) = 1 + k_2 t \xrightarrow[(1,2)]{t=1} 2 = 1 + k_2(1) \Rightarrow k_2 = 1 \end{aligned}\right\} \begin{aligned} x(t) &= 1 \\ y(t) &= t+1 \end{aligned}$$

$$\Rightarrow \int_{C_2} 2x\, ds = \int_0^1 2(1)\sqrt{0 + (1)^2}\, dt = 2\int_0^1 dt = 2t\Big|_0^1 = 2$$

$$\Rightarrow \int_C f(x,y) = 2 + \frac{5\sqrt{5}-1}{6}$$

Note: Any physical interpretation of a line integral $\int_C f(x,y)\,ds$ depends on the physical interpretation of function f. suppose that $\rho(x,y)$ represents the linear density at a point (x,y) of a thin wire shaped like curve C. Then the mass of the wire can be calculated as $\int_C \rho(x,y)\,ds$

Also, the **center of mass** of the wire w/ density function $\rho(x,y)$ is located at the point $(\bar{x},\bar{y})$ where:

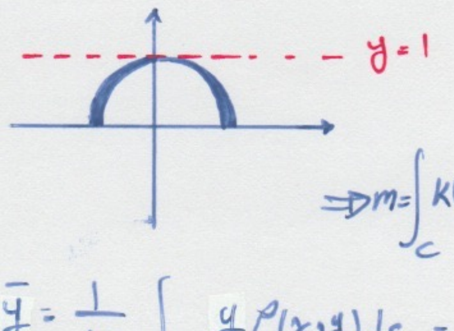
$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds \qquad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds$$

Example: A wire takes the shape of the semi-circle $x^2+y^2=1$, $y \geq 0$ and is thicker near its base than near the top. find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y=1$

Solution: Since we have a semi-circle ($y \geq 0$), the parametrization of the path would be $x = \cos t$, $y = \sin t$ and $0 \leq t \leq \pi$

$$\Rightarrow r(t) = \cos t\,\vec{i} + \sin t\,\vec{j} \Rightarrow r'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\rho(x,y) \propto (1-y) \Rightarrow \rho(x,y) = k(1-y) \Rightarrow \|r'(t)\| = \sqrt{(\sin^2 t) + (\cos^2 t)} = 1$$



we could also say $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,.dt$

$$\Rightarrow ds = dt$$

$$\Rightarrow m = \int_C k(1-y)\,ds = \int_0^\pi k(1-\sin t)\,dt = kt + k\cos t\Big]_0^\pi = k(\pi-2)$$

$$\bar{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds = \frac{1}{m}\int_0^\pi k(1-\sin t)\sin t\,.dt = \frac{k}{m}\int_0^\pi (\sin t - \sin^2 t)\,dt$$

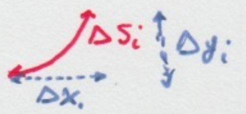
$$= \frac{k}{k(\pi-2)}\left[-\cos t - \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^\pi = \frac{4-\pi}{2(\pi-2)} \Rightarrow \bar{y} = \frac{4-\pi}{2(\pi-2)}$$

from the graph we can see $\bar{x} = 0$. why?

Note: Sometimes the line integral that we get is not in the form of $\int_C f(x,y)\,ds$. We might get the line integral in the form of

$\int_C P(x,y)\,dx + Q(x,y)\,dy$. Let's see what it means and why these integrals are really the equivalent of each other. (at the end of this section you will see a different interpretation)

We know that a small segment of C (let's say Δs_i) can be represented by the change in x and y along x-axis and y-axis



Along x-axis : $\Delta x_i = x_i - x_{i-1}$ represents Δs_i along x

Along y-axis : $\Delta y_i = y_i - y_{i-1}$ represents Δs_i along y

$$\Rightarrow \int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i = \int_a^b f(x(t), y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta y_i = \int_a^b f(x(t), y(t))\,y'(t)\,dt$$

From now on, we will frequently encounter line integrals with respect to x and y (As opposed to line integrals with respect to arc length)

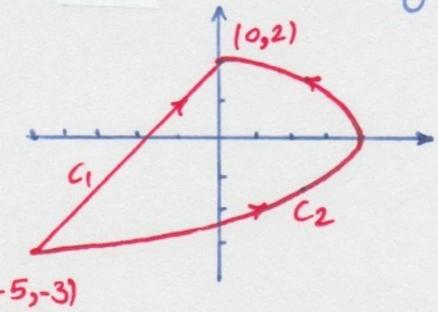
As was said above, the abbreviate form of such integrals are

$$\int_C P(x,y)\,dx + Q(x,y)\,dy$$

Note: when setting up the line integrals, the most difficult part, often is to find a parametric representation for a curve whose geometric description is given (especially when we need to parametrize a line segment).

Example: Evaluate $\int_C y^2 dx + x\,dy$ where (a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$ and (b) $C = C_2$ is the arc of parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Solution: we do not have to sketch the path (curve C) but it gives us better understanding about curve C, so we go ahead and do that.



part a) $x(t) = C_1 + k_1 t$, $y(t) = C_2 + k_2(t)$

let's say we want to choose $0 \leq t \leq 1$ and $t=0$ represents $(-5,-3)$

$\Rightarrow t=0 \longrightarrow -5 = C_1 + k(0) \Rightarrow C_1 = -5$

$-3 = C_2 + k(0) \Rightarrow C_2 = -3$

$t=1 \longrightarrow (0,2) \Rightarrow 0 = \overbrace{-5}^{C_1} + k_1(1) \Rightarrow k_1 = 5$

$2 = -3 + k_2(1) \Rightarrow k_2 = 5$

$\Rightarrow \boxed{x(t) = 5t - 5}$, $\boxed{y(t) = 5t - 3}$

Parametrization of the line segment

$$\Rightarrow \int_{C_1} y^2 dx + x\,dy = \int_0^1 \Big[(5t-3)^2 \cdot \underbrace{5 \cdot dt}_{dx} + (5t-5) \cdot \underbrace{5\,dt}_{dy}\Big] = 5\int_0^1 (25t^2 - 25t + 4)\,dt = -\frac{5}{6}$$

If we move along path C_1

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part b) we use a dummy variable like t for y

$\Rightarrow y(t) = t \longrightarrow x(t) = 4 - t^2$ $a \leq t \leq b$, Now we need to find a, and b. we know for $t=a$ $\begin{cases} x(a) = 4 - a^2 = -5 \\ x(b) = 4 - b^2 = 0 \end{cases}$ $\begin{cases} y(a) = a = -3 \\ y(b) = b = 2 \end{cases}$

$\Rightarrow a = -3$ and $b = 2 \Rightarrow -2 \leq t \leq 3$

$$\Rightarrow \int_{C_2} y^2 dx + x\,dy = \int_{-2}^{3} t^2(-2t)\,dt + (4-t^2) \cdot 1 \cdot dt = \int_{-2}^{3} (-2t^3 - t^2 + 4)\,dt = 40\frac{5}{6}$$

If we move along path C_2

Very important Note: Notice That even though the initial point and the end point of C_1 and C_2 are the same, the line integral returned two different answers when different paths are chosen. This is why we say the value of line integrals in general (we will see the exceptions later) are path dependant.

Not only the answer depends on the path, it also depends on the direction as well. e.g. if instead of C_1, we choose $-C_1$ (reverse the path) and move from $(0,2)$ to $(-5,-3)$, the line integral solution gives us $5/6$ not $-5/6$

$$\Rightarrow \int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx$$

$$\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$$

But

$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds$$

Because Δs is the arc length and it is always positive but Δx and Δy change sign when we reverse the order (reverse the direction)

line integrals in space:

The idea of line integrals in space is very similar to line integrals in plane except there are three components in line itegrals in space:

$x = x(t)$, $y = y(t)$, $z = z(t)$ $\quad a \le t \le b$

or

vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$$\boxed{\text{Also } \int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,.\,dt}$$

This can also be written as $\int_a^b f(r(t))\,|r'(t)|\,dt$

similar to line integrals in plane, we can have line integrals in space in This format: $\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$

Example: Evaluate $\int_C y \sin z\, ds$ where c is a circular helix given by $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$.

solution: $\int_C y \sin z\, ds = \int_0^{2\pi} \overset{y}{\sin t}\,.\,\overset{z}{\sin t}\,.\,\overset{ds}{\sqrt{2}\,dt}$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\,dt = \sqrt{2}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} \sin^2 t\,dt = \frac{\sqrt{2}}{2}\int_0^{2\pi}(1 - \cos 2t)\,dt = \frac{\sqrt{2}}{2}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2}\left[(2\pi - 0) - (0 - 0)\right] = \sqrt{2}\,\pi$$

Example: Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from $(2,0,0)$ to $(3,4,5)$ followed by the vertical line segment C_2 from $(3,4,5)$ to $(3,4,0)$.

Solution: the first thing that we do is to find the parametric equation for the line segment. (C_1)

$$t=0 \to (2,0,0) \longrightarrow \begin{cases} x(t)=c_1+k_1t \to \\ y(t)=c_2+k_2t \to \\ z(t)=c_3+k_3t \to \end{cases} \boxed{\begin{matrix} c_1=2 \\ c_2=0 \\ c_3=0 \end{matrix}} \quad t=1 \Rightarrow \begin{matrix} 2+k_1(1)=3 \\ 0+k_2(1)=4 \\ 0+k_3(1)=5 \end{matrix}$$

$$\boxed{k_1=1,\ k_2=4,\ k_3=5}$$

$$\Rightarrow \begin{cases} x(t)= t+2 \\ y(t)= 4t \\ z(t)= 5t \end{cases} \Rightarrow \int_0^1 4t(1)dt + 5t(4)dt + (t+2)(5dt)$$

$$= \int_0^1 (4t + 20t + 5t + 10)\,dt = \int_0^1 (29t+10)\,dt = \frac{29t^2}{2} + 10t \Big|_0^1$$

$$= 24.5$$

for line segment, C_2:

$$\begin{rcases} x(t)=c_1+k_1t \\ y(t)=c_2+k_2t \\ z(t)=c_3+k_3t \end{rcases} \quad t=0 \Rightarrow \begin{matrix} c_1=3 \\ c_2=4 \\ c_3=5 \end{matrix} \qquad t=1 \Rightarrow \begin{matrix} 3=3+k_1(1) \to k_1=0 \\ 4=4+k_2(1) \to k_2=0 \\ 0=5+k_3(1) \to k_3=-5 \end{matrix}$$

$$\Rightarrow \begin{cases} x(t)=3 \\ y(t)=4 \\ z(t)=-5t+5 \end{cases}$$

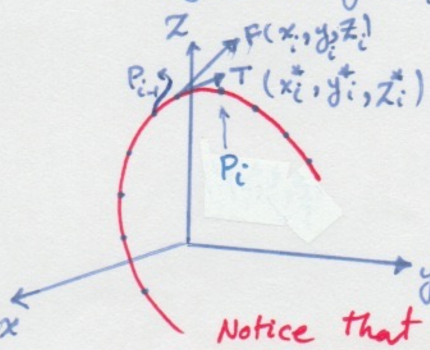
$$\Rightarrow \int_0^1 4\cdot 0\,dt + (-5t+5)\,0\,dt + (3)(-5)dt = \int_0^1 -15\,dt = -15$$

$$\Rightarrow \int_C y\,dx + z\,dy + x\,dz = 24.5 - 15 = 9.5$$

Line integrals of vector fields:

Recall that when we wanted to calculate work done by a constant force $\vec{F}$ to move an object from point P to point Q, we connected P to Q in order to have a vector like $\vec{D} = \vec{PQ}$ and then we calculated the dot product of $\vec{F}$ and $\vec{D}$: Work = $\vec{F} \cdot \vec{D}$

Now suppose that instead of a constant force, we have a vector field like F That $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + \vec{R}(x,y,z)\,\vec{k}$ and we are trying to move an object (say a particle) along a path C in this vector field.



Notice that the point (x_i^*, y_i^*, z_i^*) is b/w P_i and P_{i-1}

if we break down the curve C in smaller pieces and we pick a point between P_{i-1} and P_i which its arc length is ΔS_i. The coordinates for this point is (x_i^*, y_i^*, z_i^*). The Force vector at this point is $F(x_i^*, y_i^*, z_i^*)$ and if ΔS_i is very small. Also we assume that the tangent vector at (x_i^*, y_i^*, z_i^*)

$\Rightarrow$ the work done to move the particle from P_{i-1} to P_i in vector force field f is approximated as

$$w_i = \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \left[\Delta S_i \cdot \vec{T}(x_i^*, y_i^*, z_i^*)\right] = \underbrace{\vec{F} \cdot \vec{T}}_{\text{Dot product.}} \cdot \Delta S$$

$$\Rightarrow \text{Total work} = \sum_{i=1}^{n} F(x_i^*, y_i^*, z_i^*) \cdot T(x_i^*, y_i^*, z_i^*)\,\Delta S$$

$$\Rightarrow \int_C F(x,y,z) \cdot T(x,y,z) \cdot ds = \int_C F.T\,ds = \text{Work}$$

Therefore, work is the line integral with respect to arc length of the tangential component of the force.

we know that if the curve C is given by $r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

then $T = \frac{r'(t)}{|r'(t)|}$ $\Rightarrow$ $W = \int_a^b F(r(t)) \cdot \underbrace{\frac{r'(t)}{|r'(t)|}}_{T} \cdot \underbrace{|r'(t)| \cdot dt}_{ds} = \int_a^b F(r(t)) \cdot r'(t)\,dt$

unit tangential vector

Dot product

This integral is often written as $\int_c F.dr$

Definition: let F be a continuous vector field defined on a smooth curve C given by a vector function $r(t)$, $a \leq t \leq b$, then the **line integral of $\underline{\underline{F}}$ along $\underline{\underline{C}}$** is

$$\int_c F.dr = \int_a^b F(r(t)) \cdot r'(t)\,dt = \int_c F.T\,ds$$

Remember $F(r(t))$ is just abbreviation for $f(x(t), y(t), z(t))$ also remember $dr = r'(t)\,dt$.

Example: Find the work done by the force field $F(x,y) = x^2\vec{i} - xy\vec{j}$ in moving a particle along the quarter circle $r(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$, $0 \leq t \leq \frac{\pi}{2}$.

Solution:

since $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$

in 2D

$$F(\vec{r}(t)) = F(x(t), y(t)) = \cos^2 t\ i - \sin t \cos t\ \vec{j}$$

$$\vec{r}'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\Rightarrow \int_c F.dr = \int_0^{\frac{\pi}{2}} F(r(t)) \cdot \vec{r}'(t) \cdot dt = \int_0^{\frac{\pi}{2}} \left[-\sin t \cos^2 t - \sin t \cos t (\cos t)\right] dt$$

Dot product

$$= +\int_0^{\frac{\pi}{2}} -2\cos^2 t \sin t\,dt = \left.\frac{2\cos^3 t}{3}\right]_0^{\frac{\pi}{2}} \boxed{= -\frac{2}{3}}$$

change of var. $u = \cos t$

Note: Even though $\int_c F.dr = \int_c F.T.ds$ and integrals w/ respect to arc length are unchanged when orientation is reversed it is still true that $\int_{-c} F.dr = -\int_c F.dr$

Example: Evaluate $\int_C F.dr$, where $F(x,y,z) = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and C is the curve given by $x=t$, $y=t^2$, $z=t^3$ and $0 \leq t \leq 1$.

Solution: $r(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k} \Rightarrow r'(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$

$$\vec{F}(\vec{r}(t)) = t^3\vec{i} + t^5\vec{j} + t^4\vec{k}$$

$$\int_C F.dr = \int_0^1 F(\vec{r}(t))\, r'(t)\, dt = \int_0^1 \left[t^3 + 2t(t^5) + 3t^2(t^4)\right] dt$$

$$= \int_0^1 (t^3 + 5t^6)\, dt = \boxed{\frac{27}{28}}$$

Note: suppose that $\vec{F}$ is given on $\mathbb{R}^3$ as $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, if we try to Calculate $\int_C \vec{F}.dr$ along $C: r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$$r'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}$$

$$\Rightarrow \int_C F.dr = \int_a^b (P\vec{i} + Q\vec{j} + R\vec{k}).(x'(t)\vec{i} + y'(t)\vec{j} + z'(t)k)\, dt$$

$$= \int_a^b \left(P.x'(t) + Q\,y'(t) + R\,z'(t)\right) dt$$

we know
$x'(x)\,dt = dx$
$y'(t)\,dt = dy$
$z'(t)\,dt = dz$

$$\Rightarrow \boxed{\int_C \vec{F}.d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz \quad \text{where } F = P\vec{i} + Q\vec{j} + R\vec{k}}$$

for example $\int_C y\,dx + z\,dy + x\,dz$ in one of the previous examples can be written as $\int_C \vec{F}.d\vec{r}$ where $F(x,y,z) = y\vec{i} + z\vec{j} + x\vec{k}$